Optimización and Perturbation Analysis in the Stationary Distribution of a Markov Chain

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Abstract: **We consider a stationary distribution of a finite, irreducible, homogeneous Markov chain. Our aim is to perturb the transition probabilities matrix using approximations to find regions of feasibility and optimality for a given basis when the chain is optimized using linear programming. We also explore the application of perturbations bonds and analyze the effects of these on the construction of optimal policies.**

Key Words: **Deviation Matrix, Linear Programming, Markov Chains, Perturbation Matrix Analysis.**

I. INTRODUCTION

A perturbation in a Markov chain can be referred as a slight change in the entries of the corresponding transition stochastic matrix, resulting in structural changes in the underlying process, for example, sets of states which in the original case do not communicate, do so after a perturbation is imposed. Also, passages times that originally were not well defined random variables, may become so after the perturbation. In this sense, a square matrix is stochastic if its entries are real and non-negative and the sum of the entries in each row is equal 1.

Their importance is related with the dynamics that these represent, particularly, the singularly perturbed Markov chains have a few time scales. One time scale may correspond to the more frequent transitions occurring among states which communicate also in the unperturbed case.

In this investigation we are interested in the matrix perturbation procedure from a probabilistic point of view, where the perturbation quantity of the original stochastic matrix ϕ, can be approximated by a given matrix A such that $\phi(\epsilon) = \phi + A(\epsilon) = \phi + \epsilon A$.

Given the perturbed $\phi(\epsilon)$ matrix we approach the problem of analyzing the effects of the perturbation on the optimal policies of a Markovian decision process, sustained in the Frobenius norm of $\phi(\epsilon)$.

The Markovian process describes the productive and reproductive lifespan of herd sows, where, under an infinite planning horizon. Linear programming (LP) is used as an optimization technique.

II. THE REPLACEMENT PROBLEM

In this investigation we approach the problem of replacement management of animals in a herd, sows in this case. Regular time intervals are considered whether it should be kept a sow in the herd for an additional period or it should be replace by a new animal (gilt) and to optimize the expected return associated to the decisions made during the process, like in the investigation of Tijms [1].

Several authors have approached this problem with Markovian models or some of their variants, see for instance, Howard[2], van der Wal and Wessels[3], White and White [4], Kristensen [5] and Plá [6].

To illustrate our proposal we consider the sow replacement problem developed in Plá, Pomar and Pomar[6]. The system consist in a sow farm where sows are allowed to reach nine reproductive cycles as a maximum and at the end of each cycle, two actions can be taken: keep or replace. The problem is represented as a regular Markov decision process and solved using a linear programming model.

The transition probabilities and reward values are arbitrary but near to what are observed in actual systems; the corresponding transition probabilities matrix is perturbed using the mentioned techniques and the optimal policies are characterized in terms of these.

III. THE STOCHASTIC PROCESS NAMED MARKOV CHAIN

A stochastic process $\{M(n)\}_{n=0,1,\dots}$ with finite state space $\mathcal{Z} = \{z_1, \dots, z_S\}$ is a Markov chain with discrete time, if for all $n \in \mathbb{N}$ and all $w_0, \dots, w_n \in \mathcal{Z}$

$$\mathbf{P}\left(M(0) = w_0, M(1) = w_1, \dots, M(n) = w_n\right) =$$

$$\mathbf{P}(M(0) = w_0)\ \gamma(i, i-1),$$

where $\gamma(i, i-1) = \Pi_{i=1}^{n}\mathbf{P}(M(i) = w_i \mid M(i-1) = w_{i-1})$

Consider a Markov chain with S states $z_1, \dots, z_S$ where, in each stage $k = 1, 2, \dots$, the analyst should made a decision d, among ξ possible. Denote by $z(n) = z_i$ and $d(n) = d_k$ the state and the decision made in stage n respectively, then, the systems moves at the next stage, $n+1$, into the state z_j with perhaps, an unknown probability given by

$$\phi_{ij}^{k} = \mathbf{P}\left[z(n+1) = z_j \mid z(n) = z_i,\, d(n) = d_k\right].$$

When the transition occurs, it is followed by the reward r_{ij}^{k}, and the payoff at state z_i after the decision d_k is made is given by $\psi_i^k = \sum_{j=1}^{S} \phi_{ij}^{k}\, r_{ij}^{k}$

Since we assume that for every policy $\vartheta(k_1, \dots k_S)$, the corresponding Markov chain is ergodic, then, the steady state probabilities of this chain are given by $\phi_i^{\vartheta} = \lim_{n\to\infty} \mathbf{P}\ [Z(n) = z_i],\ \ i = 1, \dots, S$, and the problem is to find a policy ϑ for which the expected payoff

$$\Omega^{\vartheta} = \sum_{i=1}^{S} \phi_i^{\vartheta}\, \psi_i^{k}, \tag{1}$$

is maximum.

IV. OPTIMIZING MARKOV CHAIN BY LP

When the model involves an infinite horizon, the LP can be used to optimize (1), i.e., if the termination stage is unknown, usually the problem is described by an infinite planning horizon where the number N of stages is considered infinite. In this case the optimal policy is constant over stages and the objective function is given by

$$g^{\vartheta} = \sum_{i=1}^{S} \phi_i^{\vartheta} r_i^{\vartheta}, \tag{2}$$

where ϕ_i^{ϑ} is the limiting state probability under the policy ϑ (i.e., when the policy is kept constant over an infinit number of stages). This criterion maximizes the average net revenues per stage. Thus, the LP problem associated to the chain is [5]:

$$\text{maximize} \left\{ \sum_{i=1}^{S}\sum_{d=1}^{\xi} r_i^d x_i^d \ :\ \sum_{d=1}^{\xi} x_i^d - \sum_{j=1}^{S}\sum_{d=1}^{\xi} \phi_i^d x_j^d = 0, \right\}$$
$$\left\{ \sum_{i=1}^{S}\sum_{d=1}^{\xi} x_i^d = 1,\ x_i^d \geq 0, \right\} \tag{3}$$

where d is optimal in state i if and only if x_i^d from the optimal solution is strictly positive, and the x_i^d are the unconditional steady-state probabilities that the system is in the state i and decision d is made.

A replacement policy is a specification of a sequence of "keep" or "replace" actions, one for each period.

An optimal policy is a policy that achieves the greatest reward (or the smallest total net cost) of ownership over the entire planning horizon. In Pérez et al. [7] is demonstrated that the problem (3) has a degenerate solution.

V. THE APROXIMATIONS METHOD

The following questions are discussed; given the Markov chain of the problem (2), which is optimized using LP ? How affects to the optimal policy of the chain a perturbation on the optimal solution of the LP problem?

Consider the general LP problem:

$$\begin{array}{ll} \text{minimize} & f(x) = c^t\, x \\ \text{subject to} & A\, x = \delta,\ x \geq 0, \\ A_{m\times n},\ c,\ x \in \mathbb{R}^n,\ \delta \in \mathbb{R}^m & \end{array} \tag{4}$$

The number ρ of basic feasible solutions that the problem has, is less than or equal to $\binom{m}{n}$, and $\mathcal{B}_{m\times m}$ (submatrix of A) is a feasible basis of the LP model if $\mathcal{B} \in \mathcal{S}$, where $\mathcal{S} = \{\mathcal{B}_i \in A : \mathcal{B}_i^{-1}\,\delta\ \geq 0\}$.

Suppose $\mathcal{B}$ is perturbed to a matrix $\tilde{\mathcal{B}}$, that is the transition probability matrix of an n finite state, irreducible, homogeneous Markov chain as well. Denoting the stationary distribution vector of $\mathcal{B}$ by $x^\star$, and of $\tilde{\mathcal{B}}$ by $\tilde{x}$, the goal is to describe the change $(x^\star - \tilde{x})$ in the stationary distribution in terms of the changes $d\,\mathcal{B}$ using an approximations method.

In this sense, $x^\star$ and $\tilde{x}$ satisfy the systems

$$x^\star\,\mathcal{B} = x^\star, \quad x^\star > 0, \quad x^\star\,\mathbf{e} = 1$$

and

$$\tilde{x}\,\tilde{\mathcal{B}} = \tilde{x}, \quad \tilde{x} > 0, \quad \tilde{x}\,\mathbf{e} = 1$$

where $\mathbf{e}$ is the column vector of all ones.

The approximations method used can be described as follows. Given a basis $\mathcal{B} \in \mathcal{S}$, we difference the matrix equation $\mathcal{B}x = b$, and obtain, $d\mathcal{B}x + \mathcal{B}dx = 0$, i.e., $dx = -\mathcal{B}^{-1}d\mathcal{B}\,x$.

Let $d_{ij} \in d\mathcal{B}$ be the perturbation on $b_{ij} \in \mathcal{B}$, and $x^\star$ an optimal solution of the problem (4).

Defining $f^\star = f(x^\star) = c^t x^\star \leftarrow \min$, the resulting perturbation $\tilde{b}_{ij} \in \tilde{\mathcal{B}}$ can be written as

$$\tilde{b}_{ij} = b_{ij} + d_{ij}, \tag{5}$$

and therefore,

$$\tilde{x} = x^\star + dx, \tag{6}$$

constitutes a perturbated solution around of $x^\star$. Thus,

$$\tilde{f} = f(\tilde{x}) = f^\star + c^t dx, \tag{7}$$

is a new solution, not necessarily feasible (since $A\tilde{x} = \delta + Adx$) of the problem (4) evaluated in the perturbed point $\tilde{x}$. This is also an approximate solution to the modified problem:

$$\begin{array}{l} \text{minimize } f(x) = c^t\,x \\ \text{subject to } \tilde{A}\,x = \delta,\ x \geq 0, \\ \tilde{A}_{m\times n},\ c,\ x\ \in \mathbb{R}^n,\ \delta\ \in \mathbb{R}^m \end{array} \tag{8}$$

where $\tilde{A}$ is the resulting matrix after incorporating the perturbations d_{ij} in $\mathcal{B}$. Let $\hat{x}$ be an optimal solution of the problem (8), then we can write

$$\hat{x} = \tilde{x} + \varepsilon, \quad \varepsilon \in \mathbb{R}^n, \tag{9}$$

and there holds

$$\hat{f} = f(\hat{x}) = \tilde{f} + c^t\,\varepsilon\,, \tag{10}$$

The quantities, $\tilde{x} + \varepsilon$ and, $\tilde{f} + c^t\,\varepsilon$ can be viewed as approximations to $\hat{x}$ and $\hat{f}$ respectively, and ε is an error measure of the approximation. Naturally, we would want an error zero.

To evaluate the existent relationships among the ε quantity and the matrix $d\mathcal{B}$ we use the Frobenius norm $\|\cdot\|_F$ of $d\mathcal{B}$, and the Euclidian norm of ε defined as

$$\| d\mathcal{B} \|_F^2 = \text{ Trace } (d\mathcal{B}^t\,d\mathcal{B}),$$

and

$$\| \varepsilon \|^2 = (\hat{x} - \tilde{x})^t(\hat{x} - \tilde{x}), \tag{11}$$

VI. PERTURBAION BOUNDS

The norm perturbation bound used in this section is of the following form (Schweitzer [8])

$$\| x^\star - \tilde{x} \|_1 \leq \| Z \|_\infty \| d\mathcal{B} \|_\infty, \tag{12}$$

where $\| x^\star - \tilde{x} \|_1$ is the 1-norm of the vector $x^\star - \tilde{x}$ defined as the absolute entry sum, $\| \varphi \|_\infty$ is the ∞-norm of the matrix φ defined as the maximum absolute row sum, and Z is the fundamental matrix associated to the matrix $\mathcal{B}$.
Z has the form

$$Z \equiv \left[I - \mathcal{B} + \mathbf{e}\,(x^\star)^t\right]^{-1}, \tag{13}$$

Likewise, the stationary distribution vector $\tilde{x}$, of the perturbed matrix $d\mathcal{B}$ can be expressed in terms of $x^\star$ and the fundamental matrix Z as (Kemeny and Snell [9])

$$(x^\star - \tilde{x})^t = \tilde{x}^t\,d\mathcal{B}\,Z \tag{14}$$

Using (14) we formalize an important result that relates to $\hat{f}$ and $\tilde{x}$ with $f^\star$.

$$-c^t\,dx = c^t\,Z^t d\mathcal{B}^t \tilde{x}$$

equivalently

$$\tilde{f} = f^\star - c^t\,Z^t d\mathcal{B}^t \tilde{x}, \tag{15}$$

$$\hat{f} = f^\star - c^t\,[Z^t\,d\mathcal{B}^t\,\tilde{x} + \varepsilon], \tag{16}$$

To evaluate the permissible maximum value for each perturbation, we propose the alternative LP problem

$$\text{Maximize } \varphi(d) = \{d\mathbf{e} : -\mathcal{B}\,d\mathcal{B}\,dx \leq x^\star\}\,, \tag{17}$$

where $\mathbf{e} \in \mathbb{R}^\zeta$, ζ is the number of elements of the matrix $\mathcal{B}$ that will be perturbed, and $d = d_{ij}$ is the perturbations vector.

If the problem (4) has an optimal solution, then, the problem (12) also has an optimal solution because the inequality allows to slack the constrains.

To finding a feasible region φ for the perturbed basis $\tilde{\mathcal{B}}$, We define the functions $g(dx_i) = -C^t\mathcal{B}_i^{-1}d\mathcal{B}\,x^\star$, $i = 1\ldots,\rho$.

Then, a feasible region for $\tilde{\mathcal{B}}$ is given by

$$\varphi = \{d_{ij} \in g(dx_k) \,:\, g(dx_k) \leq g(dx_i), \quad i = 1, 2\ldots,\rho\}, \tag{18}$$

where the basis $\mathcal{B}_k$ used to evaluate $g(dx_k)$ is that on which the perturbation will be made.

VII. NUMERICAL EXAMPLE

Consider the following transition probabilities matrices reported in Pla et al. [7], which represent a markovian decision process with $D = 2$:

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.30 & 0 & 0.70 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & 0 & 0 & 0.75 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.20 & 0 & 0 & 0 & 0.80 & 0 & 0 & 0 & 0 & 0 \\ 0.20 & 0 & 0 & 0 & 0 & 0.80 & 0 & 0 & 0 & 0 \\ 0.20 & 0 & 0 & 0 & 0 & 0 & 0.80 & 0 & 0 & 0 \\ 0.20 & 0 & 0 & 0 & 0 & 0 & 0 & 0.80 & 0 & 0 \\ 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.75 & 0 \\ 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.75 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{d = 1 } (m \equiv \text{keep})}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{d = 2 } (r \equiv \text{replace})}$$

The corresponding LP problem is to maximize the objective function $f(y)$ given by:

$$190y_{1m} + 226y_{2m} + 232y_{3m} + 202y_{4m} + 202y_{5m} +$$
$$202y_{6m} + 202y_{7m} + 202y_{8m} + 202y_{9m} - 200B_r$$

$$y_{1m} + y_{1r} - B_m = 0,$$
$$y_{2m} + y_{2r} - 0.70y_{1m} = 0,$$
$$y_{3m} + y_{3r} - 0.75y_{2m} = 0,$$
$$y_{4m} + y_{4r} - 0.8y_{3m} = 0,$$
$$y_{5m} + y_{5r} - 0.8y_{4m} = 0,$$
$$y_{6m} + y_{6r} - 0.8y_{5m} = 0,$$
$$y_{7m} + y_{7r} - 0.8y_{6m} = 0,$$
$$y_{8m} + y_{8r} - 0.75y_{7m} = 0,$$
$$y_{9m} + y_{9r} - 0.75y_{8m} = 0,$$
$$B_r + B_m - y_{1r} - y_{2r} - y_{3r} - y_{4r} -$$
$$y_{5r} - y_{6r} - y_{7r} - y_{8r} - y_{9r} - 0.3y_{1m}$$
$$-0.25y_{2m} - 0.2y_{3m} - 0.2y_{4m} - 0.2y_{5m}$$
$$-0.2y_{6m} - 0.25y_{7m} - 0.25y_{8m} - y_{9m} = 0,$$
$$B_r + B_m + y_{1m} + y_{2m} + y_{3m} + y_{4m} +$$
$$y_{5m} + y_{6m} + y_{7m} +_{8m} + y_{9m}$$
$$+y_{1r} + y_{2r} + y_{3r} + y_{4r} + y_{5r}$$
$$+y_{6r} + y_{7r} + y_{8r} + y_{9r} = 1$$
$$y_{1m}, y_{2m}, y_{3m}, y_{4m}, y_{5m},$$
$$y_{6m}, y_{7m}, y_{8m}, y_{9m} \geq 0,$$
$$y_{1r}, y_{2r}, y_{3r}, y_{4r}, y_{5r},$$
$$y_{6r}, y_{7r}, y_{8r}, y_{9r}, B \geq 0.$$

he optimal inverse basis $\mathcal{B}^{-1}$ of the LP problem associated to this solution is

$$\begin{pmatrix} -0.78 & -0.82 & -0.82 & -0.76 & -0.69 & -0.6 & -0.48 & -0.36 & -0.21 & 0 & 0.21 \\ 0.21 & -0.82 & -0.82 & -0.76 & -0.69 & -0.6 & -0.48 & -0.36 & -0.21 & 0 & 0.21 \\ 0.14 & 0.42 & -0.57 & -0.53 & -0.48 & -0.42 & -0.34 & -0.25 & -0.14 & 0 & 0.14 \\ 0.11 & 0.31 & 0.56 & -0.40 & -0.36 & -0.31 & -0.19 & -0.11 & 0 & .11 & \\ 0.08 & 0.25 & 0.45 & 0.67 & -0.29 & -0.25 & -0.20 & -0.15 & -0.08 & 0 & 0.08 \\ 0.07 & 0.20 & 0.36 & 0.54 & 0.76 & -0.20 & -0.16 & -0.12 & -0.07 & 0 & 0.07 \\ 0.05 & 0.16 & 0.29 & 0.43 & 0.61 & 0.83 & -0.13 & -0.09 & -0.05 & 0 & 0.05 \\ 0.04 & 0.12 & 0.23 & 0.34 & 0.49 & 0.67 & 0.89 & -0.07 & -0.04 & 0 & 0.04 \\ 0.03 & 0.09 & 0.17 & 0.26 & 0.36 & 0.5 & 0.67 & 0.94 & -0.034 & 0 & 0.034 \\ 0.02 & 0.07 & 0.13 & 0.19 & 0.27 & 0.37 & 0.5 & 0.7 & 0.97 & 0 & 0.02 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

The optimal solution and the basic variables of the inverse basis are (presented in order): $B_m = 0.2106, y_{1m} = 0.2106, y_{2m} = 0.1474, y_{3m} = 0.1105, y_{4m} = 0.08847, y_{5m} = 0.07078, y_{6m} = 0.05662, y_{7m} = 0.04529, y_{8m} = 0.03397, y_{9m} = 0.02548, S_{10} = 0$. The optimal objective function is $f^\star = 163.7765$. The basis $\mathcal{B}$ that will be perturbed is formed by the columns: $y_{1m}, y_{2m}, y_{3m}, y_{4m}, y_{5m}, y_{6m}, y_{7m}, y_{8m}, y_{9m}, B_m, S_{10}$ i.e., and

$$\| d\mathcal{B} \|_F^2 = (d_{21} - 1)^2 + (d_{32} - 1)^2 + (d_{43} - 1)^2 + (d_{54} - 1)^2 +$$
$$(d_{65} - 1)^2 + (d_{76} - 1)^2 + (d_{87} - 1)^2 + (d_{98} - 1)^2 + d_{21}^2 + d_{32}^2 + d_{43}^2 +$$
$$d_{54}^2 + d_{65}^2 + d_{76}^2 + d_{87}^2 + d_{98}^2.$$

Note that the convex function $\| d\mathcal{B} \|_F^2$ achieves its minimum in $d_{ij}^\star = 0.5,\ i = 2, \ldots, 9,\ j = 1, \ldots, 8$, and $\| d\mathcal{B}^\star \|_F = 2$. In this point, $\| \varepsilon \| = 0.7280$.

Using (12) we obtain that $\| Z \|_\infty = 3.9512$, $\| d\mathcal{B} \|_\infty = 1.85$, and therefore $\| x^\star - \tilde{x} \|_1 \leq 7.3098$If x represents the optimal solution of the LP problem, then, the perturbed solution $\tilde{x} \approx x^\star - \mathcal{B}^{-1} d\mathcal{B} x$ is given by

$$\tilde{B}_m \approx 0.21 - 0.17d_{21} - 0.17d_{32} - 0.11d_{43} - 0.07d_{54} - 0.05d_{65}$$
$$-0.03d_{76} - 0.02d_{87} - 0.01d_{98}$$
$$\tilde{y}_{1m} \approx 0.21 - 0.17d_{21} - 0.17d_{32} - 0.11d_{43} - 0.07d_{54} - 0.05d_{65}$$
$$-0.03d_{76} - 0.02d_{87} - 0.01d_{98}$$
$$\tilde{y}_{2m} \approx -.9em 0.14 + 0.08d_{21} - 0.12d_{32} - 0.07d_{43} - 0.05d_{54} - 0.03d_{65}$$
$$-0.02d_{76} - 0.01d_{87} - 0.01d_{98}$$
$$\tilde{y}_{3m} \approx 0.11 + 0.06d_{21} + 0.11d_{32} - 0.05d_{43} - 0.04d_{54} - 0.02d_{65}$$
$$-0.01d_{76} - 0.01d_{87} - 0.005d_{98}$$
$$\tilde{y}_{4m} \approx 0.08 + 0.05d_{21} + 0.09d_{32} + 0.10d_{43} - 0.03d_{54} - 0.02d_{65}$$
$$-0.01d_{76} - 0.01d_{87} - 0.003d_{98}$$

$$\tilde{y}_{5m} \approx 0.07+0.04d_{21}+0.07d_{32}+0.08d_{43}+0.08d_{54}-0.01d_{65}$$
$$-0.01d_{76}-0.007d_{87}-0.003d_{98}$$
$$\tilde{y}_{6m} \approx 0.05+0.03d_{21}+0.06d_{32}+0.06d_{43}+0.06d_{54}+0.07d_{65}$$
$$-0.01d_{76}-0.005d_{87}-0.002d_{98}$$
$$\tilde{y}_{7m} \approx 0.04+0.02d_{21}+0.04d_{32}+0.05d_{43}+0.05d_{54}+0.05d_{65}$$
$$+0.06d_{76}-0.004d_{87}-0.002d_{98}$$
$$\tilde{y}_{8m} \approx 0.03+0.02d_{21}+0.03d_{32}+0.03d_{43}+0.04d_{54}+0.04d_{65}$$
$$+0.04d_{76}+0.05d_{87}-0.001d_{98}$$
$$\tilde{y}_{9m} \approx 0.02+0.01d_{21}+0.02d_{32}+0.02d_{43}+0.030d_{54}+0.03d_{65}$$
$$+0.03d_{76}+0.03d_{87}+0.04d_{98}$$
$$\tilde{S}_{10} \approx 0.94$$

For the previously developed system we use the perturbations: $d_{21} = 0.20, d_{32} = 0.20, d_{43} = 0.12, d_{54} = 0.14, d_{65} = 0.18, d_{76} = 0.10, d_{87} = 0.15, d_{98} = 0.20$; and from these, we obtain $\tilde{f} = 184.9326$, $c^t dx = 21.2314$.

Similarly, the optimal solution $\hat{x}$ of the perturbed problem is:

$$(B_m = 0.3062, y_{1m} = 0.3062, y_{2m} = 0.1531,$$
$$y_{3m} = 0.0842, y_{4m} = 0.0572, y_{5m} = 0.0377,$$
$$y_{6m} = 0.0234, y_{7m} = 0.0164, y_{8m} = 0.0098,$$
$$y_{9m} = 0.0054)$$

and $\hat{f} = 142.6643$.

Using (9) we get the ε value defined as: (-0.0160, -0.0160, -0.0303, -0.0112, -0.0019, 0.0062, 0.0141, 0.0160,0.0179, 0.0103), and the inner product $c^t\varepsilon = -42.2814$.

The Frobenius norm, the $\tilde{x} - x^\star$ norm, the ε error and other parameters were evaluated for different values of d_{ij}. In table 1 we summarize our findings and figure 1 sketch the numerical results. In the next page is the Table 2 , that shows the samples of $\hat{x}$, $x^\star$, dx and ε^2 for the proposed d_{ij}.

d_{21}	d_{32}	d_{43}	d_{54}	d_{65}	d_{76}	d_{87}	d_{98}	$\parallel d\mathcal{B} \parallel_F$	$\parallel \tilde{x} - x^\star \parallel$	$\parallel \varepsilon \parallel$	$\tilde{f}$	$\hat{f}$
0	0	0	0	0	0	0	0	0	0	0.4704	163.7765	95.1337
0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	2.5612	0.1148	0.5219	177.2500	102.0440
0.2	0.2	0.2	0.2	0.2	0.2	0.2	0.2	2.3323	0.2296	0.5842	190.7235	109.6893
0.3	0.3	0.3	0.3	0.3	0.3	0.3	0.3	2.1540	0.3443	0.6337	204.1971	240.1141
0.4	0.4	0.4	0.4	0.4	0.4	0.4	0.4	2.0396	0.4591	0.6841	217.6704	265.7273
0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	2	0.5739	0.7280	231.1442	291.1452
0.6	0.6	0.6	0.6	0.6	0.6	0.6	0.6	2.0396	0.6887	0.7633	244.6177	315.0898
0.7	0.7	0.7	0.7	0.7	0.7	0.7	0.7	2.1540	0.8035	0.7904	258.0910	335.7125
0.8	0.8	0.8	0.8	0.8	0.8	0.8	0.8	2.3323	0.9183	0.8140	271.5647	351.2073
0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9	2.5612	1.0330	0.8423	285.0382	360.8219
1	1	1	1	1	1	1	1	2.8284	1.1478	0.8835	298.5116	365.4715

Table 1: Comparative aspects of the proposed d_{ij}

Let us consider the linear programming model defined in (12). In our example it become

$$\text{maximize} = d_{21}+d_{32}+d_{43}+d_{54}+d_{65}+d_{76}+d_{87}+d_{98}$$

Subject to

$$0.1741d_{21}+0.1729d_{32}+0.1124d_{43}+0.0763d_{54}+$$
$$0.0344d_{76}+0.0208d_{87}+0.0095d_{98} \leq 0.2106$$
$$-0.0886d_{21}+0.1210d_{32}+0.0787d_{43}+0.0534d_{54}+$$
$$0.0371d_{65}+0.0180d_{76}+0.0109d_{87}+0.0050d_{98} \leq 0.1474$$
$$-0.0665d_{21}-0.1198d_{32}+0.0591d_{43}+0.0401d_{54}+$$
$$0.0278d_{65}+0.0180d_{76}+0.0109d_{87}+0.0050d_{98} \leq 0.1105$$
$$-0.0532d_{21}-0.0958d_{32}-0.1d_{43}+0.0320d_{54}+$$
$$0.0222d_{65}+0.0144d_{76}+0.0087d_{87}+0.0039d_{98} \leq 0.0884$$
$$-0.0425d_{21}-0.0766d_{32}-0.0800d_{43}-0.0848d_{54}+$$
$$0.01784d_{65}+0.0115d_{76}+0.0070d_{87}+0.0032d_{98} \leq 0.0707$$
$$-0.0341d_{21}-0.0612d_{32}-0.0641d_{43}-0.0678d_{54}-$$
$$0.0741d_{65}+0.0092d_{76}+0.0056d_{87}+0.0025d_{98} \leq 0.0566$$
$$-0.0271d_{21}-0.0490d_{32}-0.0512d_{43}-0.0542d_{54}-$$
$$0.0593d_{65}-0.0632d_{76}+0.0044d_{87}+0.0020d_{98} \leq 0.0452$$
$$-0.0204d_{21}-0.0368d_{32}-0.0384d_{43}-0.0407d_{54}-$$
$$0.0444d_{65}-0.0474d_{76}-0.0532d_{87}+0.0015d_{98} \leq 0.0399$$
$$-0.0153d_{21}-0.0275d_{32}-0.0288d_{43}-0.0304d_{54}-$$
$$0.0333d_{65}-0.0356d_{76}-0.0399d_{87}-0.04407d_{98} \leq 0.0254$$

$$d_{32} \leq 1,\ d_{43} \leq 1,\ d_{54} \leq 1,\ d_{65} \leq 1,$$
$$d_{76} \leq 1,\ d_{87} \leq 1,\ d_{98} \leq 1,$$
$$d_{ij} \geq 0, i = 2, \ldots, 9,\ j = 1, \ldots, 8$$

which solution is $d_{21} = 0.1222, d_{32} = 0.0407, d_{43} = 0.3672, d_{54} = 1, d_{65} = 1, d_{76} = 1, d_{87} = 1, d_{98} = 1, \varphi(d^\star) = 5.5302$. The corresponding Frobenius norm is $\parallel d\mathcal{B} \parallel_F = 2.6912$, and $\parallel \epsilon \parallel = 0.8688$

d_{ij}		B_m	y_{1m}	y_{2m}	y_{3m}	y_{4m}	y_{5m}	y_{6m}	y_{7m}	y_{8m}	y_{9m}
0	$\hat{x}$	0.5	0.5	0	0	0	0	0	0	0	0
	$x^\star$	0.21	0.21	0.14	0.11	0.08	0.07	0.05	0.04	0.03	0.02
	dx	0	0	0	0	0	0	0	0	0	0
	ε^2	0.08	0.08	0.02	0.01	0.01	0.05	0.003	0.002	0.001	0.001
0.4	$\hat{x}$	0.37	0.37	0.15	0.06	0.024	0.01	0.00	0.001	0.001	0.0002
	$x^\star$	0.21	0.21	0.14	0.11	0.08	0.07	0.05	0.04	0.03	0.02
	dx	−0.26	−0.26	−0.09	0.01	0.06	0.09	0.11	0.11	0.11	0.10
	ε^2	0.18	0.18	0.01	0.003	0.01	0.02	0.02	0.01	0.01	0.005
0.5	$\hat{x}$	0.33	0.33	0.16	0.08	0.04	0.02	0.01	0.005	0.002	0.001
	$x^\star$	0.21	0.21	0.14	0.11	0.08	0.07	0.05	0.04	0.03	0.02
	dx	−0.32	−0.32	−0.12	0.012	0.08	0.12	0.14	0.14	0.14	0.12
	ε^2	0.2	0.2	0.02	0.001	0.01	0.03	0.03	0.01	0.01	0.005
0.6	$\hat{x}$	0.28	0.28	0.17	0.10	0.06	0.03	0.02	0.01	0.01	0.05
	$x^\star$	0.21	0.21	0.14	0.11	0.08	0.07	0.05	0.04	0.03	0.02
	dx	−0.39	−0.39	−0.14	0.01	0.10	0.14	0.17	0.17	0.16	0.15
	ε^2	0.22	0.22	0.03	0.00	0.01	0.03	0.04	0.01	0.01	0.005
0.7	$\hat{x}$	0.23	0.23	0.16	0.11	0.08	0.05	0.04	0.02	0.01	0.01
	$x^\star$	0.21	0.21	0.14	0.11	0.08	0.07	0.05	0.04	0.03	0.02
	dx	−0.45	−0.45	−0.17	0.01	0.11	0.17	0.19	0.20	0.19	0.17
	ε^2	0.23	0.23	0.03	0.00	0.01	0.03	0.04	0.01	0.01	0.00
0.8	$\hat{x}$	0.18	0.18	0.15	0.12	0.09	0.07	0.06	0.04	0.03	0.03
	$x^\star$	0.2106	0.21	0.14	0.11	0.08	0.07	0.05	0.04	0.03	0.02
	dx	−0.52	−0.52	−0.19	0.02	0.13	0.19	0.22	0.23	0.22	0.20
	ε^2	0.25	0.25	0.04	0.00	0.01	0.03	0.04	0.01	0.01	0.01
0.9	$\hat{x}$	0.14	0.14	0.12	0.11	0.10	0.09	0.08	0.07	0.06	0.06
	$x^\star$	0.21	0.21	0.14	0.11	0.08	0.07	0.05	0.04	0.03	0.02
	dx	−0.58	−0.58	−0.22	0.02	0.15	0.21	0.25	0.26	0.25	0.22
	ε^2	0.26	0.26	0.04	0.00	0.01	0.03	0.05	0.01	0.00	0.00
1	$\hat{x}$	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
	$x^\star$	0.21	0.21	0.14	0.11	0.08	0.07	0.05	0.04	0.03	0.02
	dx	−0.65	−0.65	−0.24	0.02	0.16	0.24	0.28	0.29	0.28	0.25
	ε^2	0.29	0.29	0.03	0.00	0.02	0.04	0.05	0.01	0.01	0.01

Table 2: Samples of $\hat{x}$, $x^\star$, dx and ε^2 for the proposed d_{ij} .

VIII. CONCLUSIONS

In this document we use a method of approximations to perturb the transition probabilities matrix of a Markov chain. The Froebinous norm was used as a measure of the generated error in the approximation.

It was demonstrated that the error is minimum when $d_{ij} = 1/2$. We also obtained expressions that relate to the value of the optimal reward with the perturbed values of the gain function.

We propose an alternating model of PL to obtain the maximum bounds allowed on d_{ij} and we derive the corresponding conditions to maintain the feasibility of the problem.

In our new investigation we approach the problem assuming that the perturbations matrix $d\mathcal{B}$ can be written in terms of a Fmatrix valued function of $\mathcal{B}$ and given a matrix E we solve the following problems:

1. Approximate $F(d\mathcal{B} + E)$
2. Bound $\parallel F(d\mathcal{B} + E) - F(d\mathcal{B}) \parallel$
3. Computing the series expansion for the mean passage time matrix and for the deviation matrix of a perturbed Markov chain.

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