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Modelling systems with renewable resources based on functional operators with shift

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ABSTRACT

A method for the study of systems with renewable resources is proposed. The individual and the group parameters are separated and a discretization of time is carried out. We obtain equilibrium proportions which are functional equations with shift. A cyclic model and an open model are considered. Conditions for the existence and uniqueness of the solution are formulated for the cyclic model. For the open model, the system's evolution is analyzed.

1. Introduction

The systems whose state depends on time and whose resources are renewable form an important class of general systems. In this work, a study of the evolution of systems with one renewable resource is presented. Separation of the individual parameter and the group parameter, and discretization of time lead us to functional equations with shift. The theory of linear functional operators with shift is the adequate mathematical instrument for the investigation of such systems.

Cyclic model, where the initial state of the system coincides with the final state, is investigated. The balance equation of the cyclic model represents a lineal functional equation with shift. We formulate conditions of invertibility for the operator of the balance equation in Holder spaces with wight, in other words, we find conditions for existence and uniqueness of the equilibrium state of the system. The cyclic model is useful for the investigation of different economic and ecological problems.

Open model, where the final state of the system S does not coincide with the initial state, is considered. The state of a natural system cannot be negative. An interpretation of the negative values of the group parameter is given. The system's evolution is analyzed.

A great number of works is dedicated to systems with renewable resources, for example [1–3]. The base of the mathematical apparatus consists of differential equations in which the sought for function is dependent on time.

Our approach presupposes discretization of the processes with respect to time. In essence, we move away from the continuous tracking of changes in the system, which is to say, from a continuous time variable. A special attention is given to a detailed study of the dependence of the group parameter on the individual parameter. One example of this dependence would be the distribution of the quantity of organisms by weight in the population under consideration. For us, not only the total weight of organisms is important, but also the number of organisms of a given weight present in the system at sample time-points.

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The focus of this article is on presenting a novel approach to modelling systems with renewable resources and on applying functional operators with shift to the analysis of resulting models.

2. Conception of modelling and analysis of the system's evolution

Let S be a system with a resource λ and let T be a time interval. The choice of T is related to periodic processes taking place in the system and to human interference.

The resource λ is represented by a set of values of the individual parameter $x_i,\ i = 1, 2, \ldots, n$,

$$x_{min} = x_1 < x_1 < x_2 < \cdots < x_n = x_{max}.$$

We introduce the function $v(x_i, t)$ which expresses a quantitative estimate of the elements with the individual parameter x_i at the time t.

Let us consider a couple of examples: the system with a fish resource r and the system with a sand resource p. For fish, the eight is the individual parameter of r and $x_1, x_2, \ldots, x_{n_r}$ are the values of this individual parameter . The number of fish with a fixed weight x_i is a parameter of the group $v(x_i),\ i = 1, 2, \ldots, n_r$.

For sand, we can assign to particles a characteristic size represented by diameter. Diameter is an individual parameter of the resource p, its values are $z_1, z_2, \ldots, z_{n_p}$. The volume of the particles of the diameter z_i is a parameter of the group $\mu(z_i),\ i = 1, 2, \ldots, n_p$.

Taking time into account, the function $v(x_i, t)$ is the number of fish of the weight x_i at the time t and $\mu(z_i, t)$ is the volume of the particles of diameter z_i at the time t.

Let t_0 be the initial time and S the system under consideration.

We will follow two principles:

I. On modelling the system the description of changes that occur on the interval $(t_0, t_0 + T)$ will be substituted by fixing of the final results at the moment $t_0 + T$;
II. The principle of separation of an individual parameter x, a group parameter v and the study of dependence of v from x, $v = v(x)$.

Passing from a discrete description on to a continuous description we obtain the function $v = v(x, t),\ 0 < x < x_{max}$ which is the density of the objects of the parameter x at the time t.

The integral of the function within the limits of the individual parameters is

$$v(x_1, t) = \int_0^{x_1} v(x, t)dx, \quad v(x_i, t) = \int_{x_{i-1}}^{x_i} v(x, t)dx, \quad i = 2, 3, \ldots, n.$$

The initial state of the system S at the time t_0 is represented as the discrete distribution of the group parameter by the individual parameter

$$v(x_i, t_0) = v(x_i) \tag{1}$$

and as the density function

$$v(x, t_0) = v(x). \tag{2}$$

We will now analyze the system's evolution. In the course of time the elements of the system can change their individual parameter – fish can change their weight, and sand particles can undergo modifications.

For example, the distribution of the values of the group parameter v by the values of the individual parameter of λ at the final time $t = t_0 + T$

$$v(x_i, t_0 + T) = v\left(\frac{1}{2}x_i\right)$$

means that all of the system's elements have doubled their individual parameter relative to the initial time $t = t_0 + T$, that is, the fish grew and their weight doubled, and the particle size doubled.

In general, modifications in the distribution of the group parameters by the individual parameters is represented by a displacement. The state of the system S at the time $t = t_0 + T$ in the discrete state is

$$v(x_i, t_0 + T) = v(\beta[x_i])$$

in terms of density we have:

$$v(x, t_0 + T) = \frac{d}{dx}\beta(x)v(\beta(x)) \tag{3}$$

During the period $j_0 = [t_0, t_0 + T]$ an exception can be taken as the result of human economic activity (fishery, extraction of sand), which is represented by a summand g, so that we have a new distribution between the parameters

$$v(x_i, t_0 + T) = v[\beta(x_i)] - g(x_i)$$

in the discrete description and in the continuous case

$$v(x, t_0 + T) = \frac{d}{dx}\beta(x)v(\beta(x)) - g(x).$$

We take natural mortality into account with the coefficient d

$$v(x_i, t_0 + T) = d(x_i)v[\beta(x_i)] - g(x_i)$$

in the discreet form and in the continuous form:

$$v(x, t_0 + T) = d(x)\frac{d}{dx}\beta(x)v(\beta(x)) - g(x).$$

If natural or artificial entrance of elements to and exit of elements from the system (plant of fish, migration) have taken place, we will account for it by adding a term p.

The process of reproduction will be represented by the term $r(x)v(x)$. Thereby, the final state of the system at the moment $[t_0 + T]$ is described as follows

$$v(x_i, t_0 + T) = d(x_i)v(\beta(x_i)) + r(x)v(x) - g(x_i) + p(x_i) \tag{4}$$

in the discreet form and in the continuous form:

$$v(x, t_0 + T) = d(x)\frac{d}{dx}\beta(x)v(\beta(x)) + r(x)v(x) - g(x) + p(x). \tag{5}$$

3. Cyclic model, equilibrium proportion

Let our goal be to find an equilibrium state of the system S. That is to find such an initial distribution of the group parameter by the individual parameter $v(x_i, t_0)$ that after all transformations during the time interval $(t_0, t_0 + T))$ it would coincide with the final distribution

$$v(x_i, t_0) = v(x_i, t_0 + T), \quad i = 1, 2, \ldots, n; \tag{6}$$
$$v(x, t_0) = v(x, t_0 + T), \quad 0 < x < x_{max}. \tag{7}$$

From here, substituting Eqs. (1), (2), (4), (5) into (6), (14), it follows

$$v(x_i) = d(x_i)v(\beta(x_i)) + r(x)v(x) - g(x_i) + p(x_i) \tag{8}$$

in the discrete form and in the continuous form

$$v(x) = d(x)\frac{d}{dx}\beta(x)v(\beta(x)) + r(x)v(x) - g(x) + p(x). \tag{9}$$

Eqs. (8), (9) are called equilibrium proportions or balance equations. A model is called cyclic if the state of the system S at the initial time t_0 coincides with the state of the system S at the final time $t_0 + T$.

Rewrite the equation with discreet distributions

$$a(x_i)v(x_i) - d(x_i)v[\beta(x_i)] = f(x_i), \tag{10}$$
$$i = 1, 2, \ldots, n, \quad x_{min} = x_1 < x_1 < x_2 < \cdots < x_n = x_{max},$$

where

$$a(x_i) = 1 - r(x_i), \quad f(x_i) = p(x_i) - g(x_i)$$

and with continuous distributions

$$a(x)v(x) - b(x)v[\beta(x)] = f(x), \quad x \in [0, x_{max}], \tag{11}$$

where

$$a(x) = 1 - r(x), \quad b(x) = d(x)\beta'(x), \quad f(x) = p(x) - g(x).$$

Application of the principles I and II while modelling the systems with renewable resources leads us to functional operators with shift.

Let us make some observations on the cyclic model of the system S.

We will now clarify the appearance of the additional factor $\beta'(x)$ in (3).

Consider a group of elements whose individual parameters x would be between ς and $\xi : \varsigma \leqslant x \leqslant \xi$ at the initial time t_0, and examine its transformation in the period T. In the course of time, the individual parameters of the elements are changed and by the final moment $t_0 + T$ the individual parameters of the group well be between $\alpha(\varsigma)$ and $\alpha(\xi) : \alpha(\varsigma) \leqslant x \leqslant \alpha(\xi)$. The number of the elements of the group does not change in the period T. Therefor the integral equality is fulfilled:

$$\int_{\varsigma}^{\xi} v(x,t_0)dx = \int_{\alpha(\varsigma)}^{\alpha(\xi)} v(x,t_0+T)dx, \quad 0 \leqslant \varsigma \leqslant \xi \leqslant x_{max}. \tag{12}$$

Let the function β be the inverse of α,

$$\beta(x) = \alpha^{-1}(x)$$

After the substitution

$$z = \alpha(x), \quad x = \alpha^{-1}(z), \quad dx = \frac{d}{dz}\alpha^{-1}(z)dz,$$

in the first integral of (12) we obtain

$$\int_{\alpha(\varsigma)}^{\alpha(\xi)} v[\beta(z),t_0]\frac{d}{dz}\beta(z)dz = \int_{\alpha(\varsigma)}^{\alpha(\xi)} v(x,t_0+T)dx, \quad 0 \leqslant \varsigma \leqslant \xi \leqslant x_{max}.$$

The equality is fulfilled for each $\varsigma,\ \xi,\ 0 \leqslant \varsigma \leqslant \xi = x_{max}$; from here follows

$$v[\beta(x),t_0]\frac{d}{dx}\beta(x) = v(x,t_0+T), \quad 0 \leqslant \varsigma \leqslant \xi \leqslant x_{max},$$

and, consequently, from (2) follows (3):

$$v(x,t_0+T) = \frac{d}{dx}\beta(x)v(\beta(x)), \quad 0 \leqslant x \leqslant x_{max}.$$

Let us consider the following example. Let G be a set of organisms of one type, where the individual parameter is the weight of those organisms, and the value of the group parameter is the quantity of the organisms with the weight x,

$$x_{max} = 1, \quad \alpha(x) = x^2, \quad \varsigma = \frac{1}{3}, \quad \xi = \frac{1}{2}.$$

The function $\alpha(x) = x^2,\ x \in [0,1]$ means that the organisms which had at the time t_0 the weight x have during the period T lost their mass and at the time $t_0 + T$ will have the weight x^2. The quantity of the organisms that weighted between $\frac{1}{3} \leqslant x \leqslant \frac{1}{2}$ at the time t_0 is equal to the quantity of the organisms that weighted between $\frac{1}{9} \leqslant x \leqslant \frac{1}{4}$ at the time $t_0 + T$. Therefore

$$\int_{\frac{1}{3}}^{\frac{1}{2}} v(x,t_0)dx = \int_{\frac{1}{9}}^{\frac{1}{4}} v(x,t_0+T)dx.$$

After the substitution

$$z = x^2, \quad x = \sqrt{z}, \quad dx = \frac{1}{-}2\sqrt{z}dz$$

in the first integral, we have

$$\int_{\frac{1}{9}}^{\frac{1}{4}} v(\sqrt{z},t_0)\frac{1}{2}\sqrt{z}\ dz = \int_{\frac{1}{9}}^{\frac{1}{4}} v(x,t_0+T)dx.$$

Therefore, we obtain

$$v(\sqrt{x},t_0)\frac{1}{2\sqrt{z}} = v(x,t_0+T)$$

and

$$v(x,t_0+T) = \frac{1}{2\sqrt{z}}\ v(\sqrt{x}).$$

4. Necessary and sufficient condition of invertibility in the space of holder functions with weight

We present Eq. (11) in the operator form

$$(Av)(x) = f(x), \quad (Av)(x) \equiv a(x)(Iv)(x) - b(x)(W_\beta v)(x), \tag{13}$$

where I is the identity operator and W_β is the shift operator:

$$(Iv)(x) = v(x), \quad \big(W_\beta v\big)(x) = v[\beta(x)].$$

It is very important to know the necessary and sufficient conditions of invertibility of the operator A. They form the base for applications of different approximate methods and alow us to know when the unique solution of the equilibrium equations (13) exists. When the solution is found, the answer to the following question is known, In what initial state must the system S be that, on passing through all the transformations and having tolerated the extraction and human intervention during the period T, it would return to its previous state and would maintain its state cyclically?

In the works [4,5], the conditions of invertibility were found for the operator A in the weighted Holder space.

A function $\varphi(x)$ that satisfies the condition on the contour J,

$$|\varphi(x_1) - \varphi(x_2)| \leqslant C|x_1 - x_2|^{\mu}, \quad x_1 \in J, \ x_2 \in J, \ \mu \in (0,1),$$

is called Holder function with exponent μ and constant C on the contour J.

Let J be a segment $[0, x_{max}]$ and h be a potential function which has zeros at the endpoints $x = 0, \ x = x_{max}$:

$$h(x) = (x)^{\mu_0}(x_{max} - x)^{\mu_1}, \quad 0 < \mu_0 < 1, \ 0 < \mu_1 < 1.$$

The functions that become Holder functions and turn into zero at the points $x = 0, \ x = x_{max}$, after being multiplied by $h(x)$, form a Banach space of the Holder functions with weight h:

$$H^0_{\mu}(J, h).$$

The norm in the space $H^0_{\mu}(J,h)$ is defined by

$$\|f(x)\|_{H^0_{\mu}(J,h)} = \|h(x)f(x)\|_{H_{\mu}(J)},$$

where

$$\|h(x)f(x)\|_{H_{\mu}(J)} = \|h(x)f(x)\|_C + \|h(x)f(x)\|_{\mu},$$

and

$$\|h(x)f(x)\|_C = \max_{x \in J} |h(x)f(x)|,$$

$$\|h(x)f(x)\|_{\mu} = \sup_{x_1,x_2 \in J, x_1 \neq x_2} \frac{|h(x_1)f(x_1) - h(x_2)f(x_2)|}{|x_1 - x_2|^{\mu}}.$$

Let $\beta(x)$ be a bijective orientation-preserving displacement on the contour J: if $x_1 < x_2$ then $\beta(x_1) < \beta(x_2)$ for any $x_1 \in J, \ x_2 \in J$; and let $\beta(x)$ have only two fixed points:

$$\beta(0) = 0, \quad \beta(x_{max}) = x_{max}, \quad \beta(x) \neq x, \quad \text{when } x \neq 0, \ x \neq x_{max}.$$

In addition, let $\beta(x)$ be a differentiable function and $\frac{d}{dx}\beta(x) \neq 0, \ x \in J$.

Let functions $a(x), \ b(x)$ from the operator A belong to $H_{\mu}(J)$ and let J be [0, 1].

Invertibility conditions of a functional operator with shift in Holder spaces with weight were formulated in [4]. Here we present these conditions.

Theorem 1. *Operator A acting in the Banach space $H^0_{\mu}(J,h)$ is invertible if and only if the following condition is fulfilled:*

$$\sigma_{\beta}[a(x), b(x)] \neq 0, \quad x \in J,$$

where the function σ_{β} is defined by:

$$\sigma_{\beta}[a(x), b(x)] = \begin{cases} a(x), & \text{when } |a(i)| > [\beta'(i)]^{-\mu_i+\mu}|b(i)|, \quad i = 0,1; \\ b(x), & \text{when } |a(i)| < [\beta'(i)]^{-\mu_i+\mu}|b(i)|, \quad i = 0,1; \\ 0 & \text{in other cases}. \end{cases}$$

The proof of these conditions of existence and uniqueness of the solution to the balance equation can be found in [5]

We remark that in Lebesgue spaces with weight and in generalized Holder spaces, criteria of invertibility of the operator A were found in [6,7].

5. Open model

In this section, we do no require that the final state of the system S coincide with the initial state

$$v(x, t_0) \neq v(x, t_0 + T), \tag{14}$$

but the proposition (5)

$$v(x, t_0 + T) = d(x)\frac{d}{dx}\beta(x)v(\beta(x)) + r(x)v(x) - g(x) + p(x)$$

holds. This allows to use (5) to obtain the consequent states of the system S from the previous states.

At the time t_0 the system is in the state $v(x)$,

$$v(x) \equiv v(x, t_0).$$

At the time $t_1 = t_0 + T$, the system will be in the state $v_1(x)$:

$$v_1(x) = v(x, t_1) \equiv d(x)\frac{d}{dx}\beta(x)v(\beta(x)) + r(x)v(x) - g(x) + p(x);$$

in terms of the shift operator $(W_\beta v)(x) = v(\beta(x))$,

$$v_1(x) \equiv b(x)(W_\beta v)(x) + r(x)(Iv)(x) + f(x);$$

in terms of the operator weighted shift $(\widetilde{W}_\beta v)(x) = \beta'(x)(W_\beta v)(x)$,

$$v_1(x) \equiv d(x)(\widetilde{W}_\beta v)(x) + r(x)(Iv)(x) + f(x);$$

in terms of the operator $A =$ and $B = I - A =$,

$$v_1(x) \equiv ((I - A)v)(x) + f(x),$$
$$v_1(x) \equiv (Bv)(x) + f(x).$$

At the time $t_2 = t_0 + 2T$, the system will be in the state $v_2(x)$:

$$v_2(x) = v(x, t_2) \equiv (B(v_1 + f))(x) + f(x)v_2 \equiv B(Bv + f) + f = B^2v + Bf + f.$$

At the time $t_k = t_0 + kT$ the system will be in the state

$$v_k \equiv B(v_{k-1} + f) + f \equiv B^2v_{k-2} + Bf + f \equiv B^kv + (B^{k-1} + B^{k-2} + \cdots + B + I)f.$$

Finally, the limit state is represented by a Neumann series. It is found that over time the state of the system tends toward a limit state

$$v_\infty = \lim_{k\to\infty} u + (B^{k-1} + B^{k-2} + \cdots + B + I)f).$$

The function v_∞ presents the limit distribution of the group parameter by the individual parameter. The limit state, when $\|B\|_{H^0_\mu(J,h)} < 1$, does not depend on the initial reserve of the resource at the time $t = t_0$, $v(x, t_0)$.

The state of system S with a natural resource λ cannot be negative, but negative values of the group parameter have an interpretation for the real situation.

Consider the intervals of the individual parameter where the density function $v_k(x)$ has negative values. The appearance of such intervals with negative values of $v_k(x)$ means that the existence conditions of the system S and the human interventions are such that the resource λ cannot be sustain and decays. Elements of the system with such individual parameters are the ones that suffer most. The resource will not be able to recover in the period T. With every period, the system will pass to a state of lesser content of the resource and will therefore be running out. It is necessary to pay particular attention to the maintenance and restoration of elements with such an individual parameter that $v_k(x)$ would have negative values. To avoid the appearance of negative values it is necessary to carry out analysis of each period separately

$$[t_0, t_0 + T], [t_0 + T, t_0 + 2T], \ldots, [t_0 + (k-1)T, t_0 + kT],$$

considering how the density function is transformed

$$v_1(x), v_2(x), \ldots, v_{k-1}(x), v_k(x).$$

The integral

$$\omega(\varsigma, \xi, k) = \int_\varsigma^\xi (v_{k-1}(x) - v_k(x))dx$$

gives us the loss of what did not have time to be recovered of the fraction $\varsigma < x < \xi$ of the individual parameter in the period $[t_0 + (k-1)T, t_0 + kT]$.

It is possible to estimate at what time will the resource run out and to obtain the dynamics of the process. The analysis of the density functions allows us to take preventive measures (reduce extraction, plant elements into the system), so that the stable state v_∞ would have non-negative values.

The development of the theory of functional operators with shift is reflected in the works [8,9]. For a bibliography relating to such operators, we can point to [10,11].

The proposed approach can be applied to the modelling of more complex systems. One example would be systems with several interconnected resources, the modelling of which leads to matrix balance equations.

Precisely for the study of more complex systems would the mentioned results on functional operators with shift be useful.

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