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Slow decay of end effects in layered structures with an imperfect interface

ORLANDO AVILA-POZOS[1] and ALEXANDER B. MOVCHAN[2]
[1]*Instituto de Ciencias Básicas e Ingeniería, Universidad Autónoma del Estado de Hidalgo, Pachuca 42074, Mexico*
[2]*Department of Mathematical Sciences, University of Liverpool, Liverpool L69 3BX, UK*
(e-mail: abm@maths.liv.ac.uk)

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Abstract. An asymptotic analysis of a layered structure with an imperfect interface subject to an anti-plane shear deformation and non-homogeneous Dirichlet end conditions is presented in this paper. Two layers of isotropic materials are bonded via a middle interface layer (adhesive joint), which is thin and soft; effectively, this can be described as a discontinuity surface for the displacement. Model fields are constructed to compensate for the error produced by the asymptotic solution for the case when the layered structure is subject to non-homogeneous Dirichlet end conditions. Numerical examples and analytical estimates are presented to illustrate the slow decay of the 'boundary-layer' fields.

Key words: asymptotic analysis, boundary layer, imperfect interface

1. Introduction

Substantial progress has been made to date towards characterising the effects of imperfect interfaces for layered structures. We would like to cite papers [1, 2] on the modelling of heat-conduction problems and problems of electromagnetism for laminated composites, the work [3] on the study of anisotropic beams and papers [4, 5] presenting models of imperfect interfaces in conduction phenomena for composite structures. The papers [6, 7] discuss asymptotic models of dilute composites with imperfect interfaces and analysis of fields within a layer bonded to a substrate. Articles [8, 9] present analyses of Saint-Venant principles and developments involving nonlinear problems; end effects for problems of anti-plane shear deformation of sandwich structures are studied in [10, 11]. The Saint-Venant torsion of composite bars containing imperfect interfaces was analysed in [12] and [13] and particular attention was given to the end effects associated with certain boundary conditions.

Asymptotic analysis of thin and soft adhesive joints was presented in [14] and a further extension for orthotropic highly inhomogeneous layered structures was given in [15]. An overview of asymptotic methods used for analysis of imperfect interfaces can be found in [16].

This paper is organised as follows. In Section 2 we introduce the geometry of an isotropic highly inhomogeneous layered structure (it is a thin composite beam). It consists of an upper layer of thickness εh_1 and shear modulus μ_1 bonded via a thin and soft interface layer to a lower layer of thickness εh_2 and shear modulus μ_2, where $\varepsilon \ll 1$ is a non-dimensional positive parameter (see Figure 1).

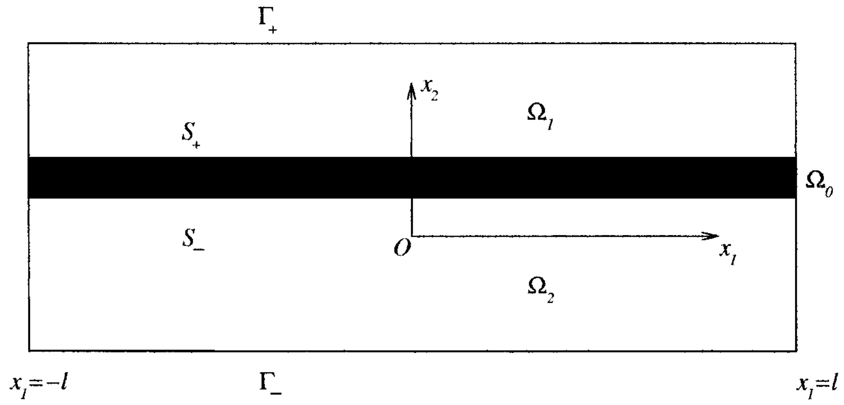

Figure 1. The two-dimensional thin region Ω_ε.

The middle layer acting like an adhesive between the adherents is *thinner* and has thickness $\varepsilon^2 h_0$, where h_0 is comparable with the thickness parameters h_1 and h_2 characterising the upper and lower layers. It is assumed that this interface is *softer* than the exterior layers: its shear modulus is defined as $\mu_0 = \varepsilon^3\mu$, where μ has the same order of magnitude as μ_1 and μ_2. In this case of special interest, it can be shown that the displacement jump across the interface is not small (it is of order $O(1)$), whereas for the same value of the interface thickness in the cases $\mu_0 = \varepsilon\mu$ and $\mu_0 = \varepsilon^2\mu$ the displacement jump would be of order $O(\varepsilon^2)$ and $O(\varepsilon)$, respectively.

In Section 3 we give the governing equations and boundary/interface conditions that describe the compound beam subject to an anti-plane shear load.

Following the idea introduced in [14], in Section 4 we study a model problem characterised by the following approximation for the displacement field $\mathbf{u}^{(i)} = (0, 0, w^{(i)}(x_1, x_2))$, $i = 0, 1, 2$

$$w^{(i)}(\mathbf{x}) \sim W^{(i,0)}(x_1, t_i) + \varepsilon W^{(i,1)}(x_1, t_i) + \varepsilon^2 W^{(i,2)}(x_1, t_i), \tag{1.1}$$

where t_i denotes a set of stretched variables (they are defined in Section 2), and the functions $W^{(1,0)}(x_1)$, $W^{(2,0)}(x_1)$ satisfy the following second-order differential equations

$$\begin{aligned}
\partial_1^2 W^{(1,0)}(x_1) &= \frac{\mu}{\mu_1 h_0 h_1}\left\{W^{(1,0)}(x_1) - W^{(2,0)}(x_1)\right\} - \frac{p^{(1)}(x_1)}{h_1},\\
\partial_1^2 W^{(2,0)}(x_1) &= -\frac{\mu}{\mu_2 h_0 h_2}\left\{W^{(1,0)}(x_1) - W^{(2,0)}(x_1)\right\} + \frac{p^{(2)}(x_1)}{h_2}.
\end{aligned}$$

It is important to remark that $W^{(1,0)}$ and $W^{(2,0)}$ do not depend upon t_1 and t_2, respectively. The functions $p^{(1)}(x_1)$ and $p^{(2)}(x_1)$ correspond to tractions on the top and on the bottom surfaces of the layered structure (see relationship (3.2)).

For the sake of simplicity we assume that a non-homogeneous Dirichlet boundary condition is applied at the left end of the thin composite beam, whereas the right end of the beam is clamped (homogeneous Dirichlet condition). In Section 5 we present further analysis of fields near the ends of the adhesive joint and derive the boundary conditions for the leading-order terms of (1.1).

In contrast with the well-known Saint-Venant theory where the end effects decay exponentially fast, for the displacement jump across the imperfect interface we have a slow decay.

In particular, when the upper and lower layers are made of the same material, with the shear modulus μ^* and the same thickness ϵh^*, the perturbation associated with the ends of the beam decays like $\exp(-\frac{\gamma}{h^*}x)$, $x > 0$,

$$\frac{\gamma}{h^*} \sim \frac{\varepsilon}{h^*}\left(\frac{2\mu h^*}{\mu^* h_0}\right)^{1/2}. \tag{1.2}$$

We note the presence of the small parameter ε in formula (1.2). Hence, this predicts a slow decay rate in comparison with the well-known behaviour for a homogeneous beam.

In Section 6 we construct the following asymptotic approximation

$$\begin{aligned} w^{(i)} \sim\ & W^{(i,0)}(x_1, t_i) + \varepsilon W^{(i,1)}(x_1, t_i) + \varepsilon^2 W^{(i,2)}(x_1, t_i) \\ & + V_-^{(i)}(\xi_1, t_i) + V_+^{(i)}(\xi_1, t_i), \qquad i = 0, 1, 2. \end{aligned} \tag{1.3}$$

The functions $V_\pm^{(i)}(\xi_1, t_i)$ (here ξ_1 is the scaled distance along the interface) compensate for the error near the edges of the layered structure with an adhesive joint, when non-zero Dirichlet conditions are prescribed at the ends of the beam.

Finally, in Section 7 we give model numerical examples that illustrate the end effects described in this paper.

2. The geometry of the sandwich beam

In this Section we define the geometry of a two-dimensional isotropic thin layered structure with an adhesive joint; this description is similar to [15]. The formulation of the problem includes two small parameters: one representing the thickness of the structure and the other corresponding to the relative softness of the adhesive. It has been shown in [14] that different relations between these parameters lead to different lower-dimensional models for the layered structure. This is qualitatively different compared to the case of a laminated structure with perfect bonding.

Let us consider a thin rectangular domain which consists of three layers as shown in Figure 1. Here

$$\Omega_1 = \{\boldsymbol{x} \in \mathbb{R}^2 : |x_1| < l\ ,\ \varepsilon(h/2 - h_1) + \varepsilon^2 h_0 < x_2 < \varepsilon h/2 + \varepsilon^2 h_0\},$$
$$\Omega_2 = \{\boldsymbol{x} \in \mathbb{R}^2 : |x_1| < l\ ,\ -\varepsilon h/2 < x_2 < -\varepsilon h/2 + \varepsilon h_2\},$$
$$\Omega_0 = \{\boldsymbol{x} \in \mathbb{R}^2 : |x_1| < l\ ,\ -\varepsilon(h/2 - h_2) < x_2 < -\varepsilon(h/2 - h_2) + \varepsilon^2 h_0\},$$

where l and h_i , $i = 0, 1, 2$, have the same order of magnitude. Also we define h as $h = h_1 + h_2$.

The interface boundary includes two parts, S_+ and S_-, specified by

$$\begin{aligned} S_+ &= \{\boldsymbol{x} : |x_1| < l\ ,\ x_2 = -\varepsilon(h/2 - h_2) + \varepsilon^2 h_0\}, \\ S_- &= \{\boldsymbol{x} : |x_1| < l\ ,\ x_2 = -\varepsilon(h/2 - h_2)\}. \end{aligned} \tag{2.1}$$

The upper and lower surfaces of the compound region are

$$\Gamma_+ = \{\boldsymbol{x} : |x_1| < l\ ,\ x_2 = \varepsilon^2 h_0 + \varepsilon h/2\},$$
$$\Gamma_- = \{\boldsymbol{x} : |x_1| < l\ ,\ x_2 = -\varepsilon h/2\}.$$

Introducing the stretched variables

$$
\begin{aligned}
t_0 &= \epsilon^{-2}(x_2 + \epsilon(h/2 - h_2) - \epsilon^2 h_0/2),\\
t_1 &= \epsilon^{-1}(x_2 - \epsilon^2 h_0 - \epsilon h_2/2),\\
t_2 &= \epsilon^{-1}(x_2 + \epsilon h_1/2),
\end{aligned} \tag{2.2}
$$

one can verify that

$$t_i \in [-h_i/2, h_i/2], i = 1, 2; t_0 \in [-h_0/2, h_0/2] \tag{2.3}$$

and

$$\partial_2 = \epsilon^{-2}\partial_{t_0}, \partial_2 = \epsilon^{-1}\partial_{t_i}, i = 1, 2, \tag{2.4}$$

where the notation ∂_α means the partial derivative with respect to x_α.

3. Formulation of the problem

Here we present a set of boundary-value problems for the Laplacian. This corresponds to the case of anti-plane shear deformation for the layered structure described in the previous Section. We look for the displacement vector which has the form

$$\mathbf{u}^{(i)} = (0, 0, w^{(i)}(x_1, x_2)),$$

with the functions $w^{(i)}$ satisfying the equations

$$\mu_i \Delta w^{(i)}(x_1, x_2) = 0. \tag{3.1}$$

Here, μ_i are the shear moduli of the materials and in all this section, the index i may take the values 0, 1, 2. We shall use the superscript index notation $w^{(i)}$ to denote the displacement in the region Ω_i.

The Neumann boundary conditions on the sides $\Gamma_\pm$ are given as follows

$$\frac{\partial w^{(1)}}{\partial x_2} = \varepsilon p^{(1)}(x_1) \text{ on } \Gamma_+, \tag{3.2}$$

$$\frac{\partial w^{(2)}}{\partial x_2} = \varepsilon p^{(2)}(x_1) \text{ on } \Gamma_-. \tag{3.3}$$

On the interfaces $S_\pm$ we assume that displacement and traction are continuous

$$w^{(1)} = w^{(0)}, \qquad \mu_1 \frac{\partial w^{(1)}}{\partial x_2} = \mu_0 \frac{\partial w^{(0)}}{\partial x_2} \quad \text{on } S_+, \tag{3.4}$$

$$w^{(0)} = w^{(2)}, \qquad \mu_2 \frac{\partial w^{(2)}}{\partial x_2} = \mu_0 \frac{\partial w^{(0)}}{\partial x_2} \quad \text{on } S_-. \tag{3.5}$$

The ends $x_1 = \pm l$ of the compound beam are assumed to be subject to the following end conditions

$$w^{(i)}(x_1 = -l, x_2) = \psi_-^{(i)}(\frac{x_2}{\epsilon}), \quad w^{(i)}(x_1 = l, x_2) = 0, \tag{3.6}$$

when $i = 1, 2$. For the middle layer ($i = 0$), we assume that the left edge is subjected to the homogeneous Neumann condition, whereas at the right end ($x_1 = l$) $w^{(0)} = 0$. Here the subscript '$-$' refers to the left end of the three-layered structure and '$+$' to the right end. The end condition at the right end is taken to be zero for the sake of simplicity without loss of generality.

Our aim is to find an asymptotic approximation, uniformly valid in the whole domain Ω_ε and its boundaries (see Figure 1), for the functions $w^{(i)}$, for $i = 0, 1, 2$ in the form (1.1).

As stated in Section 1, the middle layer is normalised in such a way that

$$\mu_0 = \varepsilon^3 \mu; \tag{3.7}$$

μ has the same order of magnitude as μ_1 and μ_2.

4. The asymptotic method far from the ends

First, we consider a special case when both ends of the beam are clamped ($\psi_-^{(i)} = 0$). Expansion (1.1) applies to the displacement outside neighbourhoods of the ends of the composite beam. Putting the series (1.1) into system (3.1) and equating the coefficients of like powers of the small parameter ε, we obtain the following recurrent system of boundary-value problems on the cross-section of the compound beam,

$$\frac{\partial^2 W^{(i,k)}}{\partial t_i^2} = -\frac{\partial^2 W^{(i,k-2)}}{\partial x_1^2} \text{ in } \Omega_i,\ i = 1, 2 \tag{4.1}$$

and

$$\frac{\partial^2 W^{(0,k)}}{\partial t_0^2} = -\frac{\partial^2 W^{(0,k-4)}}{\partial x_1^2} \text{ in } \Omega_0. \tag{4.2}$$

Also, the boundary conditions (3.2) become

$$\frac{\partial W^{(1,k)}}{\partial t_1} = \delta_{k2} p^{(1)}(x_1) \text{ on } \Gamma_+, \tag{4.3}$$

$$\frac{\partial W^{(2,k)}}{\partial t_2} = \delta_{k2} p^{(2)}(x_1) \text{ on } \Gamma_-. \tag{4.4}$$

The interface boundary conditions given by relations (3.4) and (3.5), can be written as

$$W^{(1,k)} = W^{(0,k)},\ \mu_1 \partial_{t_1} W^{(1,k)} = \mu \partial_{t_0} W^{(0,k-2)} \text{ on } S_+, \tag{4.5}$$

$$W^{(2,k)} = W^{(0,k)},\ \mu_2 \partial_{t_2} W^{(2,k)} = \mu \partial_{t_0} W^{(0,k-2)} \text{ on } S_-, \tag{4.6}$$

where $k = 0, 1, 2$, and all terms with negative indices vanish.

For convenience the following notation shall be used

$$W^{+(1,k)} = W^{(1,k)}(x_1, \frac{-h_1}{2}),\ W^{-(2,k)} = W^{(2,k)}(x_1, \frac{h_2}{2}). \tag{4.7}$$

We can easily derive the following condition for the traction on S_+ and S_- which is given by

$$\mu_2 \frac{\partial W^{-(2,2)}}{\partial t_2} = \mu_1 \frac{\partial W^{+(1,2)}}{\partial t_1} = \mu \frac{\partial W^{(0,0)}}{\partial t_0} = \frac{\mu}{h_0} D^{(0)}. \tag{4.8}$$

This relation (also see [14] and [16]) is consistent with [13, p. 235, Equation (2.6)]. Here, $D^{(0)}(x_1)$ is the *displacement jump*, which is unknown,

$$D^{(0)}(x_1) = W^{(1,0)}(x_1) - W^{(2,0)}(x_1). \tag{4.9}$$

Since $W^{(i,0)}$, $i = 1, 2$, are independent of t_i, we have

$$W^{+(1,0)} = W^{(1,0)}(x_1), \quad W^{-(2,0)} = W^{(2,0)}(x_1).$$

It can be shown that $D^{(0)}(x_1)$ must satisfy the following equation

$$h_1 h_2 \partial_1^2 D^{(0)}(x_1) - \frac{\mu(\mu_1 h_1 + \mu_2 h_2)}{\mu_1 \mu_2 h_0} D^{(0)}(x_1) = -\left(p^{(1)}(x_1) h_2 + p^{(2)}(x_1) h_1\right).$$

The leading-order terms of the asymptotic approximation (1.1) for the displacement field (3.1) satisfy the following second-order differential equations

$$\begin{aligned} \partial_1^2 W^{(1,0)}(x_1) &= \frac{\mu}{\mu_1 h_0 h_1} \left\{W^{(1,0)}(x_1) - W^{(2,0)}(x_1)\right\} - \frac{p^{(1)}(x_1)}{h_1}, \\ \partial_1^2 W^{(2,0)}(x_1) &= -\frac{\mu}{\mu_2 h_0 h_2} \left\{W^{(1,0)}(x_1) - W^{(2,0)}(x_1)\right\} + \frac{p^{(2)}(x_1)}{h_2}, \end{aligned} \tag{4.10}$$

and, when $\psi_-^{(i)} = 0$, the boundary conditions are given by

$$W^{(i,0)}(\pm l) = 0, \quad i = 1, 2. \tag{4.11}$$

It is worth mentioning that the differential equations (4.10) were derived as solvability conditions for some model boundary-value problems on the cross-section of the beam (see [15] for details). Further analysis of boundary conditions at the ends of the beam will be given in the next section.

We shall conclude this section by emphasising that $D^{(0)}(x_1)$ is of order $O(1)$. In this special case (see relation (3.7)), the interface layer can be replaced by a contour (surface) where the displacement field is discontinuous.

5. Behaviour near the ends

In this section we study a model problem near the ends of the layered structure when the end conditions are given as (3.6), where $\psi_-^{(i)} \neq 0$.

5.1. Discrepancy in the boundary conditions

The leading terms of the asymptotic series (1.1) $W^{(1,0)}$ and $W^{(2,0)}$, as specified by the system of ordinary differential equations (4.10), do not necessarily satisfy the boundary conditions at the end $x_1 = -l$. That is,

$$W^{(i,0)}(-l, t_i) \neq \psi_-^{(i)}(\varepsilon^{-1} x_2), \quad i = 1, 2.$$

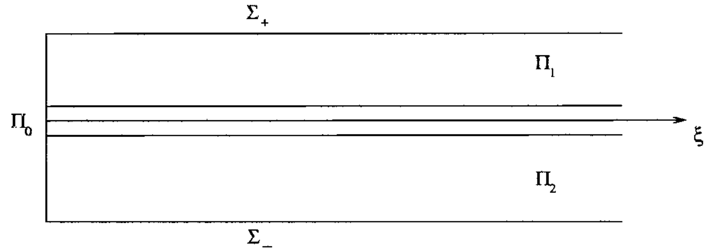

Figure 2. The semi-infinite strip.

Hence, we introduce the functions $v_-^{(i)}$ to compensate for the discrepancy near the edges of the beam. These functions will be defined as solutions of model problems of the boundary-layer type. First, we will consider the left end of the layered structure. The error to be removed is given by

$$\psi_-^{(i)}(\varepsilon^{-1}x_2) - W^{(i,0)}(-l, t_i), \quad i = 1, 2.$$

We introduce the scaled variables $\xi = \frac{x_1+l}{\varepsilon}$ and t_i, where t_1, t_2 are given by (2.2), while t_0 is re-defined as

$$t_0 = \varepsilon^{-1}(x_2 + \varepsilon(h/2 - h_2) - \varepsilon^2 h_0/2).$$

The functions $v_-^{(i)}$ satisfy Laplace's equation

$$\Delta v_-^{(i)}(\xi, t_i) = 0, \text{ in } \Pi_i, i = 0, 1, 2, \tag{5.1}$$

where $\xi > 0$ and t_i, $i = 0, 1, 2$, are specified as follows,

$$t_i \in \left[-\frac{h_i}{2}, \frac{h_i}{2}\right], \quad i = 1, 2, \qquad t_0 \in \left[-\frac{\varepsilon h_0}{2}, \frac{\varepsilon h_0}{2}\right]$$

and the regions Π_i, $i = 0, 1, 2$, (see Figure 2) are given by

$$\Pi_i = \left\{\xi > 0\,,\ -\frac{h_i}{2} < t_1 < \frac{h_i}{2}\right\}, \quad i = 1, 2,$$

$$\Pi_0 = \left\{\xi > 0\,,\ -\varepsilon\frac{h_0}{2} < t_0 < \varepsilon\frac{h_0}{2}\right\}.$$

Also, the following free-traction conditions are prescribed

$$\frac{\partial}{\partial t_1} v_-^{(1)}(\xi, t_1) = 0 \quad \text{on } \Sigma_+, \tag{5.2}$$

$$\frac{\partial}{\partial t_2} v_-^{(2)}(\xi, t_2) = 0 \quad \text{on } \Sigma_-, \tag{5.3}$$

where the upper and lower surfaces of the compound region are specified as follows

$$\Sigma_+ = \left\{\xi > 0, t_1 = \frac{h_1}{2}\right\}, \quad \Sigma_- = \left\{\xi > 0, t_2 = -\frac{h_2}{2}\right\}.$$

The Dirichlet end conditions are given by

$$v_-^{(i)}(0, t_i) = \psi_-^{(i)}(t_i) - W^{(i,0)}(-l, \frac{t_i}{\varepsilon}), i = 1, 2. \tag{5.4}$$

Here it is assumed that the materials are perfectly bonded at the interfaces Θ_+ and Θ_-,

$$v_-^{(0)}\left(\xi, \frac{\varepsilon h_0}{2}\right) = v_-^{(1)}\left(\xi, -\frac{h_1}{2}\right), \tag{5.5}$$

$$v_-^{(0)}\left(\xi, -\frac{\varepsilon h_0}{2}\right) = v_-^{(2)}\left(\xi, \frac{h_2}{2}\right), \tag{5.6}$$

$$\mu_1 \frac{\partial}{\partial t_1} v_-^{(1)}(\xi, t_1) = \mu_0 \frac{\partial}{\partial t_0} v_-^{(0)}(\xi, t_0) \text{on} \Theta_+, \tag{5.7}$$

$$\mu_2 \frac{\partial}{\partial t_2} v_-^{(2)}(\xi, t_2) = \mu_0 \frac{\partial}{\partial t_0} v_-^{(0)}(\xi, t_0) \text{ on } \Theta_-, \tag{5.8}$$

where the interfaces are given by

$$\begin{aligned} \Theta_+ &= \left\{\xi > 0, t_1 = -\frac{h_1}{2}, t_0 = \varepsilon \frac{h_0}{2}\right\}, \\ \Theta_- &= \left\{\xi > 0, t_2 = \frac{h_2}{2}, t_0 = -\varepsilon \frac{h_0}{2}\right\}. \end{aligned}$$

5.2. Exact solution

To simplify our analysis we will assume that $h_1 = h_2 \equiv h^*$ and that the materials of the upper and lower layers have the same shear modulus, $\mu_1 = \mu_2 \equiv \mu^*$. Using the separation of variables method, we write the solution of (5.1) as

$$v_-^{(i)}(\xi, t_i) = X^{(i)}(\xi) Y^{(i)}(t_i), \quad i = 0, 1, 2. \tag{5.9}$$

Hence, by using the condition

$$v_-^{(i)}(\xi, t_i) \to 0 \text{ as } \xi \to \infty,$$

it can be shown that the functions involved in the solution (5.9) are given by

$$X^{(i)}(\xi) = \mathrm{e}^{-\chi_i \xi}, \quad Y^{(i)}(t_i) = \alpha^{(i)} \cos(\chi_i t_i) + \beta^{(i)} \sin(\chi_i t_i), \tag{5.10}$$

where

$$\left(\frac{2\gamma^{(i)}}{H_i}\right)^2 = \chi_i^2 > 0$$

is a constant written in this way for convenience and the quantities H_i are given as follows

$$H_1 = H_2 \equiv h^*, H_0 = \varepsilon h_0. \tag{5.11}$$

We should note that the quantities $\gamma^{(i)}$ are some constants to be found by using conditions (5.7) and (5.8).

Using Equations (5.10), we can write the solution (5.9) as

$$v_-^{(i)}(\xi, t_i) = e^{-\chi_i \xi}(\alpha^{(i)} \cos(\chi_i t_i) + \beta^{(i)} \sin(\chi_i t_i)), i = 0, 1, 2, \tag{5.12}$$

where $\alpha^{(i)}, \beta^{(i)}, i = 0, 1, 2$ are constants to be determined from the Dirichlet end conditions (5.4).

If one takes into account the continuity of $v_-^{(i)}$ at the interface given by (5.5) and (5.6), then it follows that

$$\chi_1 = \chi_2 = \chi_0$$

and in particular, by (5.11) we obtain

$$\gamma^{(1)} = \gamma^{(2)} \equiv \gamma^{(*)}.$$

5.3. Decay rate of solution at the left end

Here we shall look for the decay rate of the function $v_-^{(i)}$ given by (5.12). Let us introduce the quantity k as follows:

$$\frac{2\gamma^{(*)}}{h^*} = \frac{2\gamma^{(0)}}{\varepsilon h_0} = \frac{2\gamma}{2h^* + \varepsilon h_0} \equiv k.$$

The exponential decay rates $2\gamma^{(i)}/H_i$ can be compared with that for a homogeneous strip of *weighted* total half-width $(h^* + \varepsilon h_0/2)$. Defining a non-dimensional weighted area fraction as

$$f = \frac{2h^*}{2h^* + \varepsilon h_0}, \tag{5.13}$$

we express $\gamma^{(*)}$ and $\gamma^{(0)}$ in terms of γ and f as follows

$$\gamma^{(*)} = \frac{f\gamma}{2}; \quad \gamma^{(0)} = \gamma(1 - f).$$

Using the boundary conditions (5.2) and (5.3) and continuity conditions (5.5)–(5.8), we may formulate the following homogeneous system of algebraic equations for the unknown coefficients $\alpha^{(i)}$ and $\beta^{(i)}$,

$$\mathbf{F}\Xi = \mathbf{0}.$$

The structure of the matrix $\mathbf{F}$ is the same as that in [10], and the vector Ξ is given by

$$\Xi = \left(\alpha_j^{(0)}, \beta_j^{(0)}, \alpha_j^{(1)}, \beta_j^{(1)}, \alpha_j^{(2)}, \beta_j^{(2)}\right)^T.$$

We seek non-trivial solutions Ξ, when the determinant of $\mathbf{F}$ vanishes. Taking into account the simplifications for this special case where the upper and lower layers are symmetric, boundary conditions (5.2) and (5.3) and continuity conditions (5.5)–(5.8), we find the following characteristic polynomial

$$\begin{aligned} &-\left(\frac{\mu^*}{\mu_0}\right)^2 \sin^2(f\gamma)\sin(2\gamma(1-f)) + \cos^2(f\gamma)\sin(2\gamma(1-f)) \\ &+\frac{2\mu^*}{\mu_0}\cos(f\gamma)\cos(2\gamma(1-f))\sin(f\gamma) = 0. \end{aligned} \tag{5.14}$$

Here, $f \neq \frac{1}{2}$ (see (5.13)). The last equation can be written simply as

$$\left(\cot(f\gamma)\cot(\gamma(1-f)) - \frac{\mu^*}{\mu_0}\right)\left(\cot(f\gamma)\tan(\gamma(1-f)) + \frac{\mu^*}{\mu_0}\right) = 0. \tag{5.15}$$

The smallest positive root of Equation (5.15) arises from the first factor. Thus, we find this root γ from the following relationship

$$\cot(f\gamma)\cot(\gamma(1-f)) - \frac{\mu^*}{\mu_0} = 0. \tag{5.16}$$

Given that the middle layer is softer than the upper and lower layers, we substitute μ_0 with $\varepsilon^3\mu$, where μ has the same order of magnitude as μ^*. Thus, the Equation (5.16) can be written as

$$\cot(f\gamma)\cot(\gamma(1-f)) = \varepsilon^{-3}\frac{\mu^*}{\mu}. \tag{5.17}$$

The quantity $1 - f$ is estimated as follows,

$$1 - f = \frac{\varepsilon h_0}{2h^* + \varepsilon h_0} \sim \frac{\varepsilon h_0}{2h^*};$$

therefore

$$f = 1 - \frac{\varepsilon h_0}{2h^*} + \cdots .$$

Also, we notice that

$$\cot\left(\frac{\varepsilon h_0}{2h^*}\gamma\right) \sim \frac{2h^*}{\varepsilon h_0 \gamma};$$

hence we find that the relationship (5.17) can be approximated as

$$\frac{1}{\gamma}\cot(\gamma\frac{2h^* - \varepsilon h_0}{2h^*}) \sim \frac{1}{\gamma}\cot(\gamma) \sim \varepsilon^{-2}\frac{\mu^* h_0}{2\mu h^*}.$$

For small values of γ, we have the following estimate

$$\cot(\gamma) \sim \frac{1}{\gamma},$$

so, finally, we can give an approximation for γ which was not discussed earlier in the literature

$$\gamma \sim \varepsilon\left(\frac{2\mu h^*}{\mu^* h_0}\right)^{1/2}. \tag{5.18}$$

Thus, we have demonstrated that the decay rate for the solution (5.12) is of order $O(1)$ since $\xi = (x_1 + l)/\varepsilon$. In this way, k predicts a slow decay rate for a layered structure with an imperfect interface.

5.4. Boundary condition for the leading-order terms

A complete solution to Equations (5.1), subject to prescribed boundary conditions at $\xi = 0$, would involve an infinite series of eigenfunctions (including the constant solution) of the form (5.12) with the following representation

$$v_-^{(i)}(\xi, t_i) = \sum_{j=0}^{\infty} \left\{ \exp(-\frac{2\gamma_j^{(i)}}{H_i}\xi) \left[\alpha_j^{(i)} \cos(\frac{2\gamma_j^{(i)}}{H_i}t_i) + \beta_j^{(i)} \sin(\frac{2\gamma_j^{(i)}}{H_i}t_i) \right] \right\}, \tag{5.19}$$

for $i = 0, 1, 2$. Here the coefficients $\alpha_j^{(i)}, \beta_j^{(i)}, i = 0, 1, 2;\ j \geq 1$ are the same as in [11].

To find the value of the coefficient $\alpha_0^{(i)}$ in each layer, one has to use the end conditions (5.4). Evaluating the solution of the problem in Equations (5.19) at the left end, we have

$$\begin{aligned} v_-^{(i)}(0, t_i) &= \alpha_0^{(i)} + \sum_{j=1}^{\infty} \left[\alpha_j^{(i)} \cos\left(\frac{2\gamma_j^{(i)} t_i}{H_i}\right) + \beta_j^{(i)} \sin\left(\frac{2\gamma_j^{(i)} t_i}{H_i}\right) \right] \\ &= \psi_-^{(i)}(t_i) - W^{(i,0)}(-l), i = 1, 2. \end{aligned}$$

Integrating with respect to the scaled cross-section variable t_i in each layer, one finds that

$$\int (\psi_-^{(i)}(t_i) - W^{(i,0)}(-l))\mathrm{d}t_i = \int \alpha_0^{(i)} \mathrm{d}t_i.$$

The functions $v_-^{(i)}$ of the boundary-layer type vanish at infinity if and only if $\alpha_0^{(i)} = 0$, and hence, when $i = 1, 2$,

$$W^{(i,0)}(-l) = \frac{1}{H_i} \int \psi_-^{(i)}(t_i)\mathrm{d}t_i, \tag{5.20}$$

which constitutes the left-end conditions for the terms $W^{(i,0)}$, $i = 1, 2$.

Analogously, one can derive the following condition at the right end by using the homogeneous Dirichlet boundary condition (3.6),

$$W^{(i,0)}(+l) = 0. \tag{5.21}$$

6. Compensating functions and a uniform asymptotic approximation

Following Section 5, we introduce the functions named as *compensating functions*. They have the form

$$V_-^{(i)}(x_1, t_i) = \sum_{j=1}^{\infty} c_j \exp(-\frac{2\hat{\gamma}_j^{(i)}}{H_i}(x_1 + l))Y_j^{(i)}(t_i), \tag{6.1}$$

where $\gamma_j^{(i)} = \varepsilon \hat{\gamma}_j^{(i)}$ is the normalised exponent (see (5.18)), and c_j are constants to be specified. It is emphasised that $\int Y_j^{(i)} \mathrm{d}t_i = 0$, when we integrate over the cross-section and $Y_j^{(i)}$ is given by (5.10).

Equally, the analysis for the right end of the thin layered structure suggests including some other functions $V_+^{(i)}$, $i = 0, 1, 2$ with the following structure:

$$V_+^{(i)}(x_1, t_i) = -\sum_{j=1}^{\infty} d_j \exp(-\frac{2\hat{\gamma}_j^{(i)}}{H_i}(l - x_1))Y_j^{(i)}(t_i). \tag{6.2}$$

The functions $V_-^{(i)}$ and $V_+^{(i)}$ can be regarded as a result of a multiple-reflection effect from the ends of the layered structure due to the fact of the slow decay rate of the functions described in Subsection 5.3.

To find the constants involved in this analysis, namely $c_j, d_j, j \geq 1$, we look for the combination of $V_-^{(i)} + V_+^{(i)}$ satisfying the following relationships at the edges

$$(V_-^{(i)} + V_+^{(i)})|_{x_1=-l} = \sum_{j\geq 1} Y_j^{(i)}(t_i), \quad (V_-^{(i)} + V_+^{(i)})|_{x_1=l} = 0. \tag{6.3}$$

After evaluating the functions (6.1) and (6.2) at the ends and substituting them in the system (6.3), we get a system for the constants c_j and d_j

$$c_j - d_j \exp(-\frac{4\hat{\gamma}_j^{(i)} l}{H_i}) = 1, \quad c_j \exp(-\frac{4\hat{\gamma}_j^{(i)} l}{H_i}) - d_j = 0,\ j \geq 1.$$

The solution for the constants gives

$$c_j = \frac{1}{1 - \exp(-\frac{8\hat{\gamma}_j^{(i)} l}{H_i})}; \quad d_j = \frac{\exp(-\frac{4\hat{\gamma}_j^{(i)} l}{H_i})}{1 - \exp(-\frac{8\hat{\gamma}_j^{(i)} l}{H_i})}. \tag{6.4}$$

Finally we are able to construct a uniform asymptotic approximation for the functions $w^{(i)}$, which is given by

$$\begin{aligned} w^{(i)} \sim\ & W^{(i,0)}(x_1) + \varepsilon W^{(i,1)}(x_1) + \varepsilon^2 W^{(i,2)}(x_1, t_i) \\ & + V_-^{(i)}(x_1, t_i) + V_+^{(i)}(x_1, t_i), \quad i = 0, 1, 2. \end{aligned} \tag{6.5}$$

We recall that the leading-order terms $W^{(i,0)}(x_1)$, $i = 1, 2$ are the functions that satisfy the boundary-value problem (4.10)–(4.11) on the cross-section. In the last relationship, $V_-^{(i)} + V_+^{(i)}, i = 0, 1, 2$, compensate for the discrepancy left by $W^{(1,0)}(x_1)$ and $W^{(2,0)}(x_1)$ at the left end. The so-called compensating functions model the interaction between the ends when non-zero conditions are prescribed along the left edge.

7. Numerical examples and final remarks

In this Section a simple numerical example, programmed in PDEToolBox of MATLAB, illustrates a very slow decay rate of the boundary layer for a layered structure with an imperfect interface.

The outer layers were considered to have the same thickness

$$\varepsilon h_1 = \varepsilon h_2 = 0{\cdot}6,$$

whereas the middle layer is thinner

$$\varepsilon^2 h_0 = 0{\cdot}06.$$

The length of the region was taken to be $2l = 3$. The elastic materials of the regions $\Omega_i, i = 0, 1, 2$, are characterised by Young's moduli E_i and by the values ν_i of the Poisson ratio, $i = 0, 1, 2$.

For all the tests we considered the upper material to be brass with the following elastic moduli

$$E_{+(\mathrm{Br})} = 100 \text{ GPa};\ \nu_{+(\mathrm{Br})} = 0{\cdot}25.$$

The lower material was assumed to be made of aluminium, with the following elastic moduli

$$E_{-(Al)} = 70 \text{ GPa};\ \nu_{-(Al)} = 0{\cdot}30.$$

Table 1. The end conditions

Left end	Right end
$w^{(1)} = 1{\cdot}5$	$w^{(1)} = 0$
$w^{(2)} = -2$	$w^{(2)} = 0$
$\partial_{x-1} w^{(0)} = 0$	$w^{(0)} = 0$

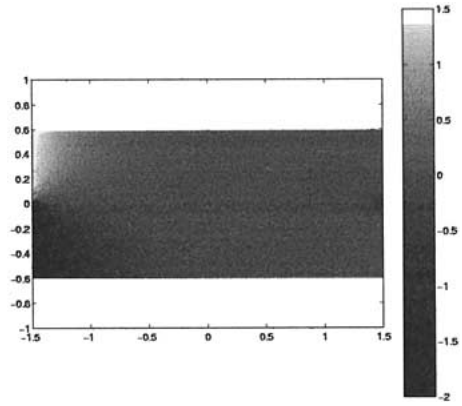

Figure 3. The displacement w corresponding to the case when the middle interface layer is stiff (it is made of CFRP); it illustrates a fast decay of the displacement discontinuity far away from the left end of the beam.

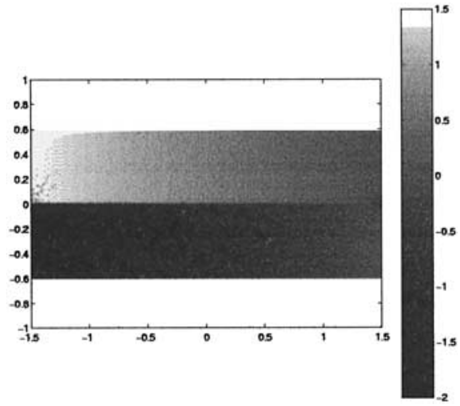

Figure 4. The displacement w corresponding to the case when the middle adhesive layer is soft (it is made of Scotweld AF-6); the displacement discontinuity is seen along the whole length of the beam, as predicted by the asymptotic theory.

We present the test consisting of homogeneous Neumann boundary conditions applied on Γ_+ and Γ_- and the edges conditions as shown in Table 1.

For further details of the elastic parameters for adherents and adhesives we refer to the monograph [17].

Case 1, A thin and stiff middle layer
In this case, it is assumed that E_0 is comparable with E_1 and E_2. The middle layer was assumed to be made of CFRP (Carbon Fibre Reinforced Laminates) (see Figure 3). This material is characterised by the elastic moduli,

$$E_{\text{CFRP}} = 135\ \text{GPa};\ \nu_{\text{CFRP}} = 0{\cdot}30.$$

Case 2, A thin and soft middle layer
In this case we considered a thin middle layer made of Scotweld AF-6 (see Figure 4) which is characterised by the elastic moduli,

$$E_{\text{Pf}} = 0{\cdot}07\ \text{GPa};\ \nu_{\text{Pf}} = 0{\cdot}49.$$

The observation of the numerical results shows that, when we have a thin and stiff middle layer, the decay rate of the end conditions is very fast as shown in Figure 3, so that we can easily compare this result with the structure of two perfectly bonded layers.

In contrast, in Figure 4 a slow decay of the boundary-layer fields is observed. This is the case when the thin middle layer is acting as an adhesive and gives the interesting effect of a 'multiple reflection' throughout the layered beam.

The expansion (6.5) describes the approximation of the displacement field within a thin rectangular layered structure including a soft and thin middle layer subjected to an anti-plane shear load.

A future development is envisaged in the analysis of 'boundary-layer' fields for torsion problems involving bars with imperfect interfaces (also see [13]) as well as the analysis of the edge effects for layered plates with imperfect interfaces.

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