Growth Functions of Finitely Generated Algebras

Brandon George, Luis Angel Herrera Flores, and Brittany Schlomer
Advisor: Dr. Harold Ellingsen

Throughout $F = \mathbb{R}$ or $\mathbb{C}$ and $0 \in \mathbb{N}$.

Definition 1. *Let A be a vector space over F equipped with an additional associative binary operation from $A \times A$ to A, denoted here by $\cdot$ (i.e. if x and y are any two elements of A, $x \cdot y$ is the product of x and y). Then A is an algebra over F (an F-algebra) if the following hold for all elements x, y, and z in A, and all elements a and b in F:*

- $(x + y) \cdot z = x \cdot z + y \cdot z$
- $x \cdot (y + z) = x \cdot y + x \cdot z$
- $(ax) \cdot (by) = (ab) \cdot (xy)$.

Definition 2. *Let A be an F-algebra. We say that A is finitely generated provided there is $\{a_1, a_2, \cdots, a_r\} \subseteq A$ such that every element of A can be written as a finite linear combination of monomials in $a_1, a_2, \ldots, a_r$. V will denote the F-span of $\{a_1, a_2, \ldots, a_r\}$. V is called a finite dimensional generating subspace (fdgs) for A.*

Definition 3. *Let A be an F-algebra with finite dimensional generating subspace $V = span\{a_1, a_2, \ldots, a_r\}$. The length of a monomial in A is the number of letters that make up the monomial, counting repetitions. Define $V^0 = F$ and for $n \geq 1$, V^n as the F-span of monomials in $a_1, \ldots, a_r$ of length n and $A_n = \sum_{i=0}^{n} V^i$.*

Proposition 1. *Let V^n, for $n \geq 0$, be the F-span of monomials in $a_1, \ldots, a_r$ of length n, and $A_n = \sum_{i=0}^{n} V^i$. Then*

$$A = \bigcup_{n=0}^{\infty} A_n$$

Proof. "$\supseteq$" Since each $A_n \subseteq A$ for all $n \in \mathbb{N}$, $\bigcup_{n=0}^{\infty} A_n \subseteq A$.
"$\subseteq$" Let $a \in A$. Then since $\{a_1, a_2, \ldots, a_r\}$ is a generating set for A, a can be written as a finite linear combination of monomial in $a_1, \ldots, a_r$. Let k be the maximum length of these monomials. Then $a \in A_k \subseteq \bigcup_{n=0}^{\infty} A_n$, so $A = \bigcup_{n=0}^{\infty} A_n$. $\square$

Also, note that for the A_n's as defined above, $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ is an ascending chain of finite dimensional spaces.

Definition 4. *Define a growth function of A with respect to V $d_V : \mathbb{N} \to \mathbb{N}$ by $d_V(n) = dim(A_n) = dim(\sum_{i=0}^{n} V^i)$. We note that in general $dim\sum_{i=0}^{n} V^i \neq \sum_{i=0}^{n} dim V^i$*

We would like to know what types of functions these growth functions can be. Are they polynomials or exponential functions?

Example 1. *What is a growth function for $\mathbb{R}[x]$, the commutative polynomial algebra in one variable? Let $n \in \mathbb{N}$.*
It has fdgs $V = span\{x\}$. Each $V^n = span\{x^n\}$, so $\{x^n\}$ is a basis for V^n. Since $\{1, x, \ldots, x^n\}$ is a basis for polynomials of at most degree n, $d_V(n) = dim(A_n) = n + 1$.

Example 2. *What is a growth function for $\mathbb{R}[x, y]$, the commutative polynomial algebra in two variables? Let $n \in \mathbb{N}$.*
It has fdgs $V = span\{x, y\}$. Each basis element of V^n will be of the form $x^a y^b$, where $a + b = n$. There are $n+1$ choices for a and one corresponding b for each a, so each V^n will have $n+1$ basis elements. Thus $d_V(n) = \sum_{i=0}^{n}(i+1) = \frac{n^2+3n+2}{2}$.

Example 3. *What is a growth function for $\mathbb{R}\langle x, y\rangle$, the free algebra in two variables? Let $n \in \mathbb{N}$.*
Note that x and y do not commute. $\mathbb{R}\langle x, y\rangle$ has fdgs $V = span\{x, y\}$. Each V^n has 2^n basis elements since there are 2 choices for each letter of a monomial of length n. Thus $d_V(n) = \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

Proposition 2. *Every finitely generated algebra is isomorphic to a quotient of a finitely generated algebra.*

Proof. Let A be a finitely generated algebra with generating set $\{a_1, a_2, \ldots, a_r\}$. Define $\psi : F\langle x_1, x_2, \ldots, x_r\rangle \to A$ by setting $\psi(x_i) = a_i$ for $1 \leq i \leq r$ and extending to $F\langle x_1, x_2, \ldots, x_r\rangle$ in the natural way. ψ is a surjective algebra homomorphism. By the First Isomorphism Theorem, we see that $F\langle x_1, x_2, \ldots, x_r\rangle/\ker(\psi) \simeq A$. □

Note that every ideal of $F\langle x_1, x_2, \ldots, x_r\rangle$ is the kernel of some surjective homomorphism, so in order to calculate growth functions for various algebras, we may calculate them for quotients of finitely generated free algebras. In particular, we will look at quotients whose ideals are generated by finitely many monomials in $x_1, x_2, \ldots, x_r$. We will refer to monomials as <u>words</u> and denote them by $m_1, m_2, \ldots, m_k$. An ideal generated by the set $\{m_1, m_2, \ldots, m_k\}$ is the set of linear combinations of words who contain at least one $m_1, m_2, \ldots, m_k$ as a factor (subword) denoted $I = (m_1, \ldots, m_k)$. Such ideals are called <u>monomial ideals</u>.

Throughout I will be a monomial ideal with generators $m_1, m_2, \ldots, m_k \in F\langle x_1, \ldots, x_r\rangle$ and $A = F\langle x_1, x_2, \ldots, x_r\rangle/I$. A will be called a <u>finitely generated, finitely presented monomial algebra</u>. In a quotient algebra, we can view words in I as zero, so every element of A can be written as a linear combination of words not in I. Let $\mathcal{B}$ be the collection of words not in I including 1. $\mathcal{B}$ is a spanning set for A. $\mathcal{B}$ consists of the words that do not have any of $m_1, m_2, \ldots, m_k$ as a subword.

Proposition 3. *$\mathcal{B}$ is a basis for A.*

Proof. We have already seen that $\mathcal{B}$ is a spanning set. Let $w_1, w_2, \ldots, w_k \in \mathcal{B}$. Suppose $\alpha_1 w_1 + \cdots + \alpha_t w_t$ is zero for some $\alpha_1, \ldots, \alpha_t \in F$. By zero, we mean an element of I, $\alpha_1 w_1 + \cdots + \alpha_t w_t \in I$. None of $w_1, w_2, \ldots, w_t$ is in I, so
$\alpha_1 = \alpha_2 = \cdots = \alpha_t = 0$. Thus we have spanning and independence, therefore we have a basis. □

Throughout, let $V = \text{span}\{x_1, x_2, \ldots, x_r\}$ is a fdgs for A, so $V^n =$ the span of words in $\mathcal{B}$ of length n. So dim $V^n =$ number of words in $\mathcal{B}$ of length n. Since $A_n = \sum_{i=0}^{n} V^i$ and $\mathcal{B}$ is a basis for A, dim$A_n =$ the number of words in $\mathcal{B}$ of length at most n. Hence calculating a growth function is counting words.

Example 4. *Determine a growth function for $\mathbb{R}\langle x, y\rangle/I$ where $I = (xy)$.*

n	***Words in $\mathcal{B}$ of length n***
0	1
1	x, y
2	x^2, y^2, yx
3	x^3, y^3, y^2x, yx^2

Let b_n denote the number of words in $\mathcal{B}$ of length n. Then $b_n = dimV^n$. The growth function $d_V(n) = \sum_{i=0}^{n} b_i$. Since $I = (xy)$, any word with xy as a subword is zero. Given $n \geq 1$, there is only one word of length n in $\mathcal{B}$ beginning with x namely x^n. There are n such words beginning with y, namely $y^k x^{n-k}$ for $1 \leq k \leq n$.So there are $n+1$ words of length n in $\mathcal{B}$, i.e., $dimV^n = n+1$. Thus, $d_V(n) = \sum_{i=0}^{n} (i+1) = \frac{n^2+3n+2}{2}$. Note the growth function is a quadratic polynomial, and is identical to that for the commutative polynomial in two variables. However these algebras are not isomorphic.

Proposition 4. *Any subword of a word in $\mathcal{B}$ is also in $\mathcal{B}$.*

Proof. Suppose $w \in \mathcal{B}$ and v is subword of w that is not in $\mathcal{B}$. Since $v \notin \mathcal{B}$, there is a generator of I that is subword of v. Hence w has a generator of I as a subword. □

Note, the longer words in $\mathcal{B}$ are made up of shorter words in $\mathcal{B}$. The words of length d determine words of length greater than d.

Example 5. $I = (x^2y, y^2x)$
The word $w = xy^2x^2y^3$ has subwords of length 3: $xy^2, y^2x, yx^2, x^2y, xy^2, y^3$. Thus $w \notin \mathcal{B}$ because it contains the subwords x^2y and y^2x which are in I.

We need a better way to count our words. It turns out that this better way is using a <u>directed graph</u>.

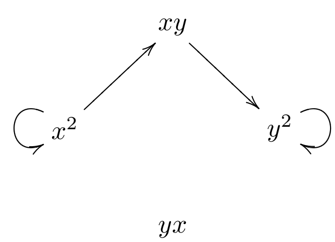
Definition 5. *A <u>directed graph</u> is a set V of vertices with a set E of ordered pairs of vertices called <u>arrows</u>.*

Definition 6. *Let u, v be words. We say u is a prefix of v provided there is a word w for which $v = uw$. We say u is a suffix of v provided that there is a word z for which $v = zu$.*

Example 6. *Let $w = x^2y^3x = (x^2y)(y^2x) = (x^2y^2)(yx)$. x^2y is a prefix of w and yx is a suffix of w.*

Let $d+1$, where $d \geq 2$, be the maximum length of the generators in I and $\{w_1, w_2, \ldots, w_k\}$ be words in $\mathcal{B}$ of length d. We use this set of words as vertices for a directed graph. We draw an arrow from w_i to w_j provided there is a word in $\mathcal{B}$ of length $d+1$ whose prefix of length d is w_i and whose suffix of length d is w_j. We will call our graph an overlap graph for $\mathcal{B}$, and denote it by Γ. Equivalently there are letters a and b such that $w_i a = b w_j \in \mathcal{B}$.

Example 7. $I = (yx^2, y^2x, xyx, yxy)$
$d + 1 =$ *maximum length of generators in* $I = 3$
$d =$ *max length*$-1 = 2$.
d is the length of the vertices: x^2, y^2, xy, yx
$x^2 \to xy$ provided there is a word of length 3 in $\mathcal{B}$ whose prefix is x^2 and suffix is xy.
Words of length 3 in $\mathcal{B}$: x^3, y^3, x^2y, xy^2



Definition 7. *A path in a directed graph is a sequence of arrows in the same direction. We call path $u_1 \to u_2 \to \cdots \to u_t \to u_1$ a cycle provided $u_i \neq u_j$ for $i \neq j$. The length of a path is the number of arrows in it.*

Example 8.

path	***word***
$x^2 \to xy$	x^2y
$x^2 \to xy \to y^2$	x^2y^2
$yx \to x^2 \to xy \to y^2 \to yx$	yx^2y^2x

number of arrows	***length of word***
1	3 $(d+1)$
2	4 $(d+2)$
4	6 $(d+4)$

Proposition 5. *A path in Γ with j arrows, for $j \geq 1$, corresponds to a unique word in $\mathcal{B}$ of length $d+j$. A word in $\mathcal{B}$ of length $d+j$ corresponds to a unique path in Γ with j arrows.*

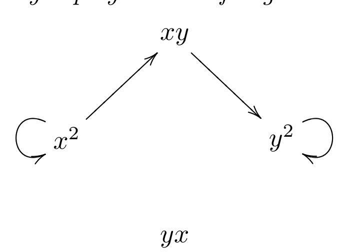
Proof. Let $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{j+1}$ be a path in Γ of length j, where $v_i \in \{w_1, \ldots, w_t\}$. Denote $v_k^* =$ suffix of v_{k+1} of length $d-1$ for $k = 1, 2, \ldots, j$. Since $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_{j+1}$ is a path, we can write $v_i = v_{i-1}^* a_{i-1}$, where a_{i-1} is a single letter for $i = 2, 3, \ldots, j+1$. The corresponding word is $w = v_1 a_1 a_2 \ldots a_j$, which is of length $d+j$. Since the generators of I are of max length $d+1$, we only need to look at the subwords of w of length $d+1$. The subwords of length $d+1$ are each determined by $v_i \rightarrow v_{i+1}$, where $1 \leq i \leq j$, and each of these corresponds to a word in $\mathcal{B}$ by definition, and hence, do not contain any generators of I as a subword.

Let $w \in \mathcal{B}$ with length $d+j$. Write $w = y_1 y_2 \ldots y_{d+j}$ and consider the subwords of length d, $v_i = y_i y_{i+1} \ldots y_{i+d-2} y_{i+d-1}$, $1 \leq i \leq j$. Since $v_{i+1} = y_{i+1} y_{i+2} \ldots y_{i+d-1} y_{i+d}$, we have $v_i \rightarrow v_{i+1}$ when $1 \leq i \leq j$. Thus, w corresponds to the path $v_1 \rightarrow \cdots \rightarrow v_{j+1}$.

□

Theorem 1 (Ufnarovski)**.** *Consider the monomial algebra $F\langle x_1, \ldots, x_r\rangle / I$ where I is generated by finitely many monomials. Let $d+1$ be the maximum length of the monomials that generate I. Let Γ denote the overlap graph for $\mathcal{B}$, the basis of words for A with vertices the words of length d in $\mathcal{B}$.*
(1) If Γ has two intersecting cycles, then the growth function for A is exponential.
(2) If Γ has no intersecting cycles, then the growth function for A is bounded above and below by two polynomials of degree s where s is the maximal number of distinct cycles on a path in Γ.

Example 9. *Recall graph in Example 7.*
$I = (yx^2, y^2x, xyx, yxy)$
$d+1 =$ *maximum length of generators in* $I = 3$
$d =$ *max length*$-1 = 2$.
vertices: x^2, y^2, xy, yx
The overlap graph has no intersecting cycles and has two cycles on a path, so the growth function is bounded by a polynomial of degree 2.



It is known that growth functions for algebras are either exponential or polynomial. We would like to know more specifically, for a given d, what types of growth functions are attainable. We can construct an algebra by first costructing an overlap graph. We begin with all words of length d and connect v to w when the suffix of length $d-1$ of v is the prefix of length $d-1$ of w. Arrows not drawn are words in I.

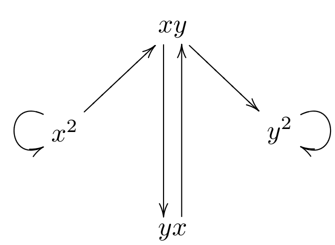
Proposition 6. *For some ideal I generated by words of at most length $d+1$, the corresponding algebra $F\langle x, y\rangle/I$ has exponential growth.*

Proof. Consider $I = (y^{d+1})$. Then the following cycles intersect: $x^d \to x^d$ and $x^d \to x^{d-1}y \to x^{d-2}yx \to x^{d-3}yx^2 \to \cdots \to yx^{d-1} \to x^d$. So by Ufnarovski's Theorem, $F\langle x, y\rangle/I$ has exponential growth. $\square$

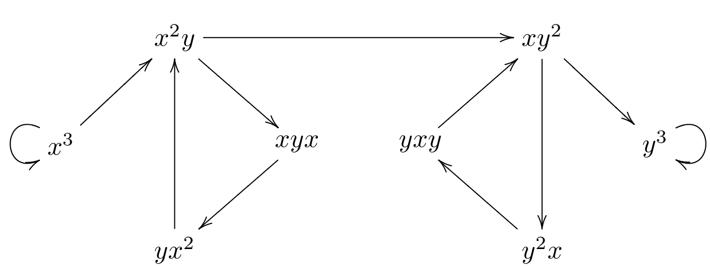
So we know that for any d, exponential growth will be possible. Next we will look at what degrees are possible when the algebras have polynomial growth.

Conjecture 1 (Dr. Ellingsen's Conjecture)**.** *If I is generated by words of at most length $d+1$, then the growth function is either exponential or is polynomial with degree at most $d+1$.*

We have shown for $d = 2$ that the growth function must be either exponential or bounded by a polynomial of degree at most 3.



Additionally, we have shown that for $d = 3$, the growth function must be either exponential or bounded by a polynomial of degree at most 4.



What about $d = 4$?

$$
\begin{tikzcd}
 & yx^3 \arrow[dddd] & & & x^2y^2 \arrow[lll] & & & y^3x \arrow[dl] & \\
 & & xyx^2 \arrow[dr] & & yxyx \arrow[dd] & & y^2xy \arrow[dd] & & \\
x^4 \arrow[loop left] \arrow[ddr] & & & yx^2y \arrow[dl] & & xy^2x \arrow[uul] & & & y^4 \arrow[loop right] \\
 & & x^2yx \arrow[uu] \arrow[rr] & & xyxy \arrow[uu] \arrow[rr] & & yxy^2 \arrow[dr] & & \\
 & x^3y \arrow[ur] \arrow[rrr] & & & x^2y^2 \arrow[uur] & & & xy^3 \arrow[uuuu] \arrow[uur] &
\end{tikzcd}
$$

$I = (yx^4, xyx^3, yxyx^2, y^2x^2y, yx^2y^2, x^2y^3, yxy^2x, y^2xyx, y^3x^2, xy^2xy, y^4x)$

The conjecture fails for $d = 4$. We would like to find a new maximum or upper bound on the degrees of growth functions for a given d. A theorem by Dr. Ellingsen gives us a starting point.

Theorem 2 (Ellingsen)**.** *If there are $d + i$ words of length d, the growth function is either exponential or bounded by a polynomial of degree $i + 1$.*

This gives us a really high upper bound on the possible degrees for our growth functions. There are 2^d words of length d, which we can write as $d + (2^d - d)$ words, so the growth of our algebra with corresponding ideal generated by words of length at most $d+1$ is either exponential or bounded by a polynomial of degree $2^d - d + 1$.

Although $d + 1$ is not a maximum for a growth function, we can show that growth functions up to degrees $d + 1$ are attainable. To prove this we need a few more definitions.

Definition 8. *Let v be a word of length p and w a word of length $d \geq p$. w is periodic provided w is a prefix of v^j from some positive integer j. We call v a base for w and the length p is the period for w. The smallest possible period is the minimal period.*

Example 10. *1.) Let $w = x^2yx^2yx$. Then w has minimal period 3 with base x^2y. Note that w also has period 6 with base x^2yx^2y.*
2.) Let $u = x^2yx^2$. Interestingly u has periods 3 and 4 with bases x^2y and x^2yx respectively.

Definition 9. *Let $w = a_0a_1 \ldots a_{d-1}$ be a word of length d. Then any word of the form $a_ia_{i+1} \ldots a_{d-1}a_0 \ldots a_{i-1}$ is called a cyclic permutation of w.*

Note that we can draw an arrow from any word to exactly one cyclic permutation of itself, namely $a_0a_1 \ldots a_{d-1} \to a_1a_2 \ldots a_{d-1}a_0$.

Example 11. *Let $w = xy^2xy$. Then the cyclic permutations of w are $xy^2xy, y^2xyx, yxyxy, xyxy^2, yxy^2x, xy^2xy$. Note these all connect and give us a cycle: $xy^2xy \to y^2xyx \to yxyxy \to xyxy^2 \to yxy^2x \to xy^2xy$.*

Lemma 1. *Let w be a word of length d. Every vertex on a cycle of length $p < d$ must be periodic with minimal period of length p. Moreover, the bases of length p for any two words on these cycles are cyclic permutations of each other.*

Proof. FINISH □

Proposition 7. *Let w be a word of length d with minimal period d. Then w and its cyclic permutations form a cycle of length d.*

Proof. Let $w_0 = w$. We know that the cyclic permutations of w are on a path of length d that begins and ends at w_0: $w_0 \to w_1 \to \cdots \to w_{d-1} \to w_0$ where $w_1, w_2, \ldots, w_{d-1}$ are cyclic permutations of w. Suppose that this path does not form a cycle. In other words, some of our vertex words are equal. We will show that w_0 is on a cycle of length less than d. Consider the set of ordered pairs of equal words, $U = \{(w_i, w_j) | w_i = w_j \text{ and } i < j\}$. Choose i_1 the least index of the first coordinates of the elements of U. Choose j_1 the largest index such that $(w_{i_1}, w_{j_1}) \in U$. We begin our path $w_0 \to w_1 \to \cdots \to w_{i_1} = w_{j_1}$ (if $i_0 = 0$, this is just the vertex w_0). If there is no $(w_i, w_j) \in U$ with $i > j_1$, then $w_0 \to w_1 \to \cdots \to w_{i_1} \to w_{j_1+1} \to \cdots \to w_{d-1} \to w_0$ is a cycle of length at most $d-1$. Suppose there is an ordered pair $(w_i, w_j) \in U$ with $i > j_1$. Then choose i_2 to be the least index greater than j of the first coordinates and choose j_2 the greatest index such that $(w_{i_2}, w_{j_2}) \in U$. Then we extend the path: $w_0 \to \cdots \to w_{i_1} \to w_{j_1+1} \to \cdots \to w_{i_2} = w_{j_2}$. By choice of i_1 and i_2, each of these words are distinct. Again, if there is no $(w_i, w_j) \in U$ with $i > j_2$, we have that $w_0 \to \cdots \to w_{i_1} \to w_{j_1+1} \to \cdots \to w_{i_2} \to w_{j_2+1} \to \cdots \to w_{d-1} \to w_0$ is a cycle. Otherwise, we repeat this process. Since our i_k's are strictly increasing and less than $d-1$, this will terminate, leaving us with a cycle of length smaller than d. By the preceding lemma, w has period smaller than d, a contradiction to minimal period d. □

Let $[u]$ denote all cyclic permutations of the word u. We will write $[u] \to [v]$ to denote that some cyclic permutation of u connects to some cyclic permutation of v.

Proposition 8. *For some ideal I generated by words of length at most $d+1$, the corresponding algebra has growth function of degree $d+1$.*

Proof. Consider the path $[x^d] \to [x^{d-1}y] \to [x^{d-2}y^2] \to \cdots \to [x^2y^{d-2}] \to [xy^{d-1}] \to [y^d]$. We have cycles of length 1 at x^d and y^d. Let $1 \le i \le d-1$. The word $x^{d-i}y^i$ has minimal period d. By the proposition, $x^{d-i}y^i$ and its cyclic permutations form a cycle of length d. Note the number of x's in each of these words is $d-i$. This yields $d+1$ cycles on a path. Since words on different cycles have different number of x's, the cycles are disjoint.

□

Additionally, any degree smaller than $d+1$ can be attainable by removing the arrows corresponding to any number of cycles along the above path. We would like to know the maximum possible degree that is attainable for $d=4$. We can do this by putting as many distinct cycles on a path as possible by using the smallest cycles first. For $d=4$, there are $2^4=16$ possible vertices to use in cycles. We want to start by finding all the cycles which contain only one vertex, namely, x^4 and y^4. By exhaustion, we can find all cycles containing 2, 3, and 4 vertices.

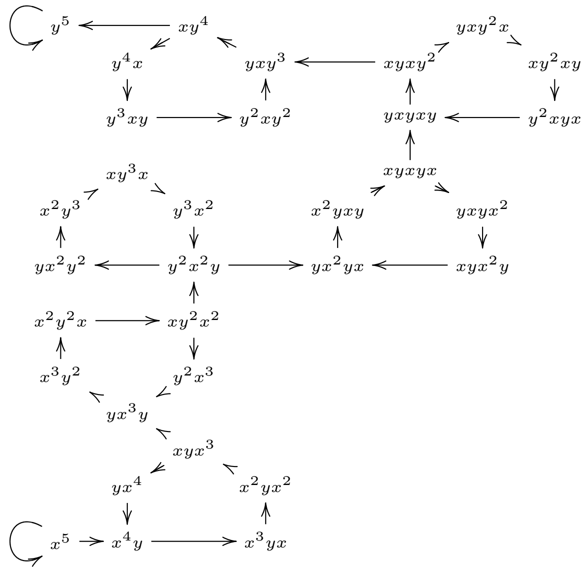
Number of vertices in a cycle	Number of cycles
1	2
2	1
3	2
4	3

By using two cycles with 1 vertex, one cycle with 2 vertices, two cycles with 3 vertices, and one cycle with 4 vertices, we use 14 out of the total 16 possible vertices $1(2)+2(1)+3(2)+4(1)=14$. We cannot include another cycle with 4 vertices because we would use some vertices twice which would give us exponential growth. Thus, we could potentially connect these 6 cycles in a path which would correspond to a maximum possible degree of 6 for the growth function. This is just a possible upper bound and is not the actual path as shown previously. Our previous example in fact shows 6 is the upper bound.

Consider $d=5$

Number of vertices in a cycle	Number of cycles
1	2
2	1
3	2
4	3
5	≥ 4

Similarly to the $d=4$ case, we can count the number of distinct cycles that we can put in a path using only $2^5=32$ vertices. $1(2)+2(1)+3(2)+4(3)+5(2)=32$. This gives us an upper bound of 10 cycles.



Conjecture 2. *For d prime, there are $\frac{2^d-2}{d}$ cycles of length d.*

Example 12. *For $d = 5$, we have $\frac{2^5-2}{5} = 6$ cycles of length 5. We have connected all 6 cycles of length 5 on a path.*

Example 13. *For $d = 3$, note d is prime, the path with $\frac{2^2-2}{2} + 2 = 4$ cycles is $x^3 \to [x^2y] \to [xy^2] \to y^3$.*

For $d = 5$, the path with $\frac{2^3-2}{3} + 2 = 8$ cycles is
$x^5 \to [x^4y] \to [x^3y^2] \to [x^2y^3] \to [x^2yxy] \to [xy^2xy] \to [xy^4] \to y^5$

We have also shown this is true for $d = 7$ for a path with $\frac{2^7-2}{7} + 2 = 20$ cycles.

Conjecture 3. *For d prime, all of the cycles of length d can be connected on a path.*

Proposition 9. *Let d be an odd integer. For some ideal I generated by words of length at most $d+1$, the corresponding algebra has growth function of degree $2d-2$.*

Proof. Consider the path $[x^{d-i}y^i] \to [x^{d-i-2}y^ixy] \to [x^{d-i-1}y^{i-1}] \to [x^{d-i-1}y^{i+1}]$. We have an arrow $[x^{d-i}y^i] \to [x^{d-i-2}y^ixy]$ because $x^{d-i-1}y^ix \in [x^{d-i}y^i]$ and $x^{d-i-1}y^ix \to x^{d-i-2}y^ixy$. We have an arrow $[x^{d-i-2}y^ixy] \to [x^{d-i-1}y^{i-1}]$ because $y^ixyx^{d-i-2} \in [x^{d-i-2}y^ixy]$ and $y^{i-1}xyx^{d-i-2} \in [x^{d-i-1}y^{i-1}]$ and $y^ixyx^{d-i-2} \to y^{i-1}xyx^{d-i-2}$. Finally, we have an arrow $[x^{d-i-1}y^{i-1}] \to [x^{d-i-1}y^{i+1}]$ because $xyx^{d-i-1}y^{i-1} \in [x^{d-i-1}y^{i-1}]$ and $yx^{d-i-1}y^i \in [x^{d-i-1}y^{i+1}]$ and $xyx^{d-i-1}y^{i-1} \to yx^{d-i-1}y^i$. For $i = 2, 4, \ldots, d-3$, these paths connect end to end since we always have an arrow $x^{d-i-1}y^{i-1} \to x^{d-i-2}y^{i+2}$. Consider the path $[x^d] \to [x^{d-1}y] \to [x^{d-2}y^2] \to \cdots \to [x^2y^{d-2} \to [xy^{d-1}] \to [y^d]$ where the middle consists of the previously described paths of length 4 connected end to end. This path contains $2d-2$ cycles. Take I to be the ideal containing the words corresponding to all arrows not on this path. $\square$

Additionally, we see that growth functions of any degree less than $2d-2$ is attainable by removing any combination of cycles from the path we constructed in our proof.