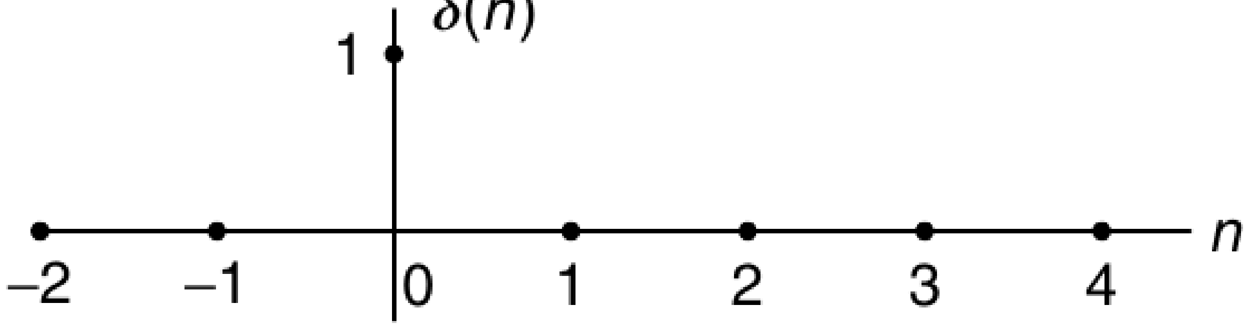
Digital signals and systems

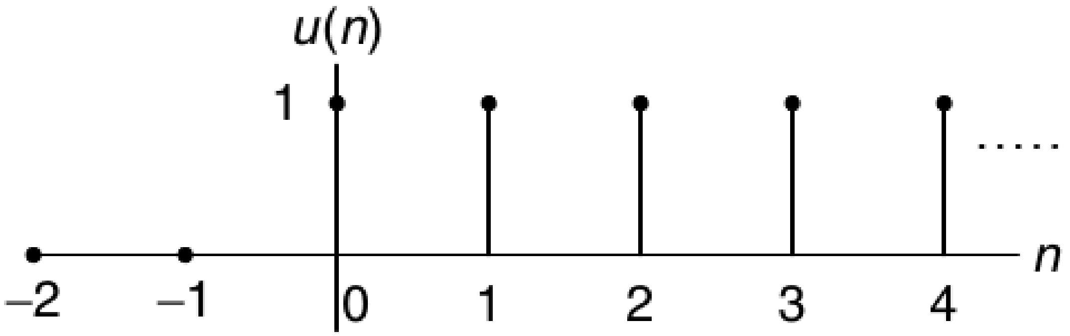
Unit impulse sequence

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

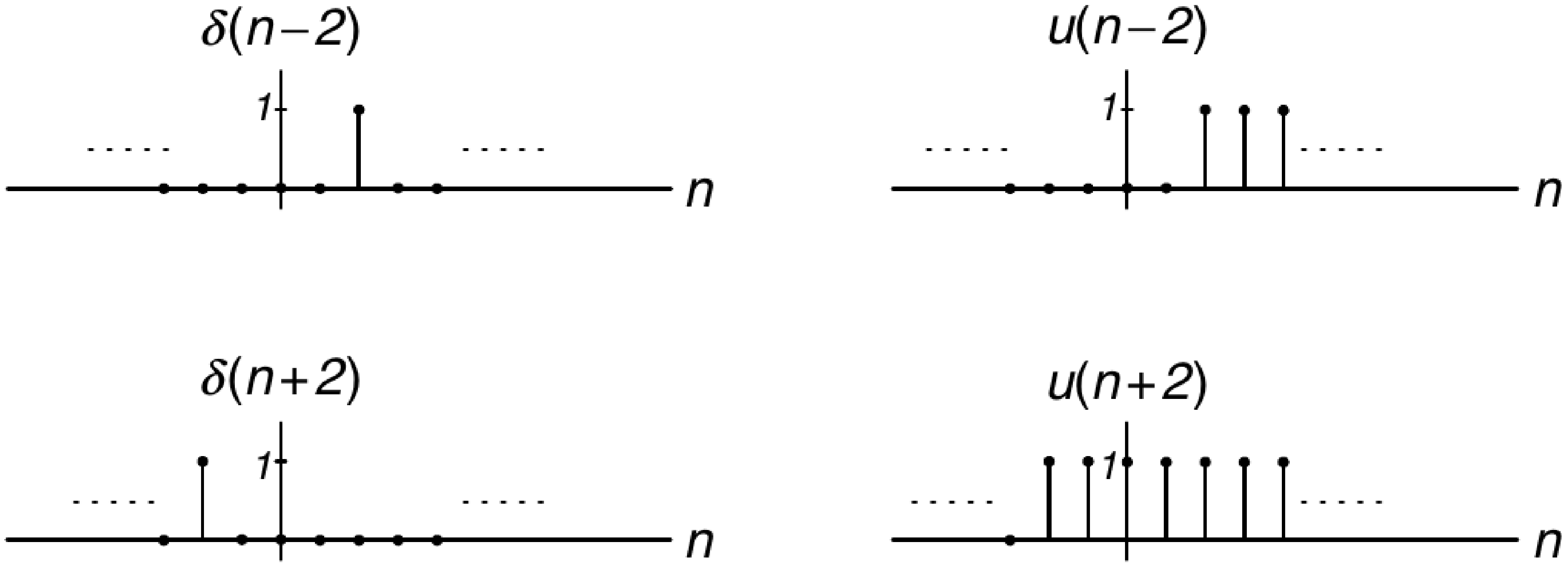


Unit step sequence

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



Shifted sequences

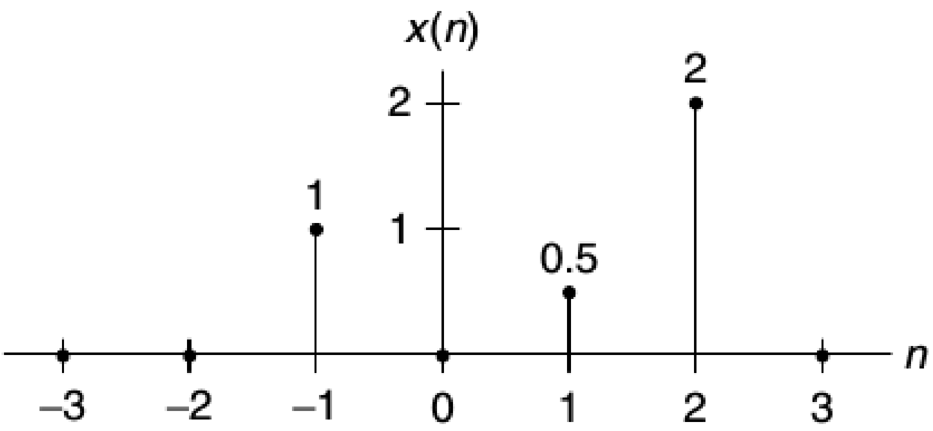


Example

Given the following,

$$x(n) = \delta(n+1) + 0.5\delta(n-1) + 2\delta(n-2),$$

a. Sketch this sequence.



3.1. Sketch each of the following special digital sequences:

a. $5\delta(n)$

b. $-2\delta(n-5)$

c. $-5u(n)$

d. $5u(n-2)$

3.2. Calculate the first eight sample values and sketch each of the following sequences:

a. $x(n) = 0.5^n u(n)$

b. $x(n) = 5\sin(0.2\pi n)u(n)$

c. $x(n) = 5\cos(0.1\pi n + 30^0)u(n)$

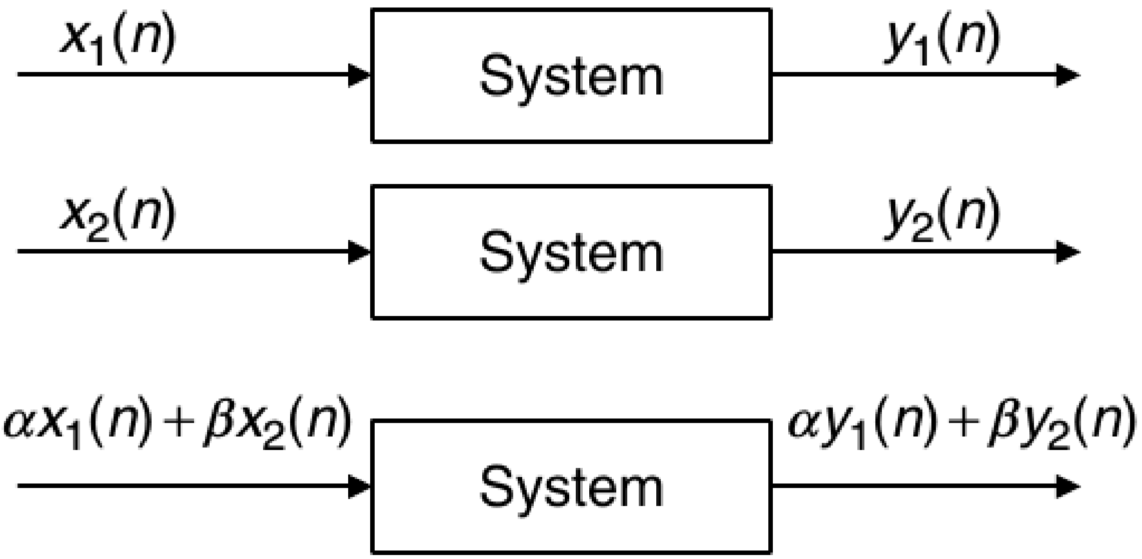
d. $x(n) = 5(0.75)^n \sin(0.1\pi n)u(n)$

3.3. Sketch the following sequences:

a. $x(n) = 3\delta(n+2) - 0.5\delta(n) + 5\delta(n-1) - 4\delta(n-5)$

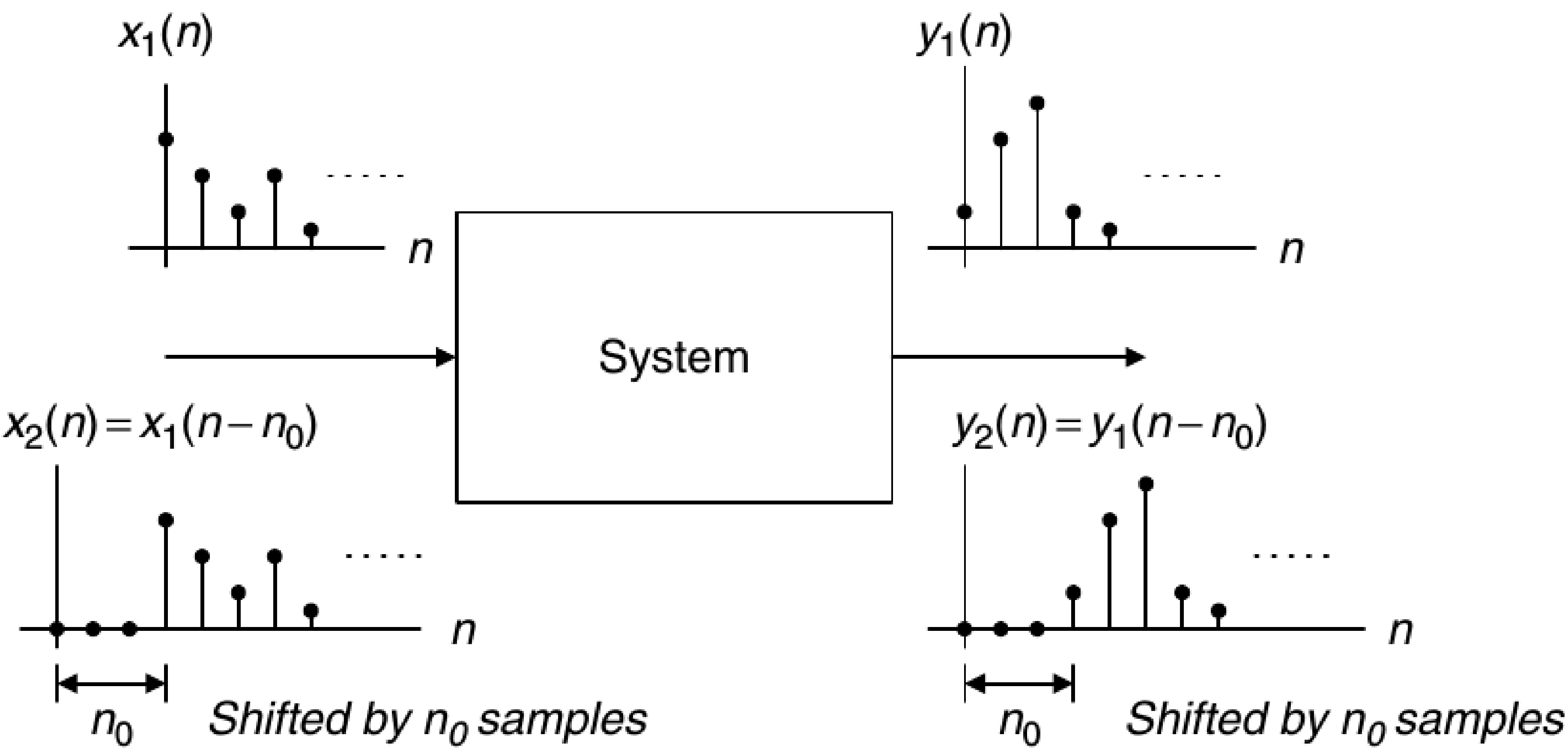
b. $x(n) = \delta(n+1) - 2\delta(n-1) + 5u(n-4)$

Linearity



$x_1(n)$
System
$y_1(n)$
$x_2(n)$
System
$y_2(n)$
$\alpha x_1(n)+\beta x_2(n)$
System
$\alpha y_1(n)+\beta y_2(n)$

Time invariance



Causality

- A causal system is one in which the output $y(n)$ at time n depends only on the current input $x(n)$ at time n, its past input sample values such as $x(n-1)$, $x(n-2)$, . . . : Otherwise, if a system output depends on the future input values, such as $x(n+1)$, $x(n+2)$, . . . , the system is noncausal.

3.6. Determine which of the following is a linear system.

a. $y(n) = 5x(n) + 2x^2(n)$

b. $y(n) = x(n-1) + 4x(n)$

c. $y(n) = 4x^3(n-1) - 2x(n)$

3.7. Given the following linear systems, find which one is time invariant.

a. $y(n) = -5x(n-10)$

b. $y(n) = 4x(n^2)$

3.8. Determine which of the following linear systems is causal.

a. $y(n) = 0.5x(n) + 100x(n-2) - 20x(n-10)$

b. $y(n) = x(n+4) + 0.5x(n) - 2x(n-2)$

Diference equations

- A causal, linear, time-invariant system can be described by a difference equation having the following general form:

$$y(n) + a_1 y(n-1) + \ldots + a_N y(n-N)$$

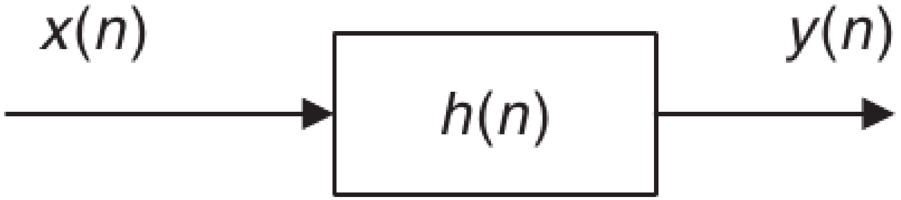
$$= b_0 x(n) + b_1 x(n-1) + \ldots + b_M x(n-M),$$

$$y(n) = -\sum_{i=1}^{N} a_i y(n-i) + \sum_{j=0}^{M} b_j x(n-j).$$

System Representation Using Its Impulse Response

A linear time-invariant system can be completely described by its unit-impulse response, which is defined as the system response due to the impulse input d(n) with zero initial conditions

With the obtained unit-impulse response h(n), we can represent the linear time-invariant system



Example

Given the linear time-invariant system $y(n) = 0.5x(n) + 0.25x(n-1)$ with an initial condition $x(-1) = 0$,

- Determine the unit-impulse response h(n).
- Draw the system block diagram.
- Write the output using the obtained impulse response.

Solution:

a. According to Figure 3.13, let $x(n) = \delta(n)$, then

$$h(n) = y(n) = 0.5x(n) + 0.25x(n-1) = 0.5\delta(n) + 0.25\delta(n-1).$$

Thus, for this particular linear system, we have

$$h(n) = \begin{cases} 0.5 & n = 0 \\ 0.25 & n = 1 \\ 0 & elsewhere \end{cases}$$

b. The block diagram of the linear time-invariant system is shown as

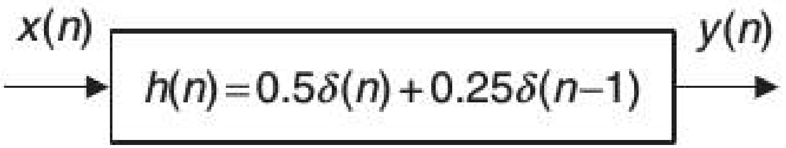


FIGURE 3.15 **The system block diagram in Example 3.7.**

c. The system output can be rewritten as

$$y(n) = h(0)x(n) + h(1)x(n-1).$$

Convolution
(Digital convolution sum)

$$y(n) = \ldots + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots.$$

$$h(n) = \ldots + h(-1)\delta(n+1) + h(0)\delta(n) + h(1)\delta(n-1) + h(2)\delta(n-2) + \ldots,$$

Example

Given the difference equation

$y(n) = 0.25y(n-1) + x(n)$ for $n \geq 0$ and $y(-1) = 0$,

- Determine the unit-impulse response $h(n)$.
- Draw the system block diagram.
- Write the output using the obtained impulse response.
- For a step input x(n)=u(n), verify and compare the output responses for the first three output samples using the difference equation and digitial convolution sum

Solution:

a. Let $x(n) = \delta(n)$, then

$$h(n) = 0.25h(n-1) + \delta(n).$$

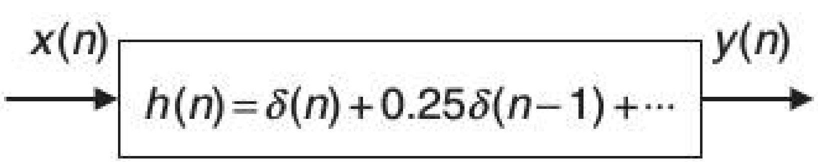
To solve for $h(n)$, we evaluate

$$h(0) = 0.25h(-1) + \delta(0) = 0.25 \times 0 + 1 = 1$$
$$h(1) = 0.25h(0) + \delta(1) = 0.25 \times 1 + 0 = 0.25$$
$$h(2) = 0.25h(1) + \delta(2) = 0.25 \times 0.5 + 0 = 0.0625$$
$$\cdots$$

With the calculated results, we can predict the impulse response as

$$h(n) = (0.25)^n u(n) = \delta(n) + 0.25\delta(n-1) + 0.0625\delta(n-2) + \ldots.$$

b. The system block diagram is given in Figure 3.16.



c. The output sequence is a sum of infinite terms expressed as

$$\begin{aligned} y(n) &= h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots \\ &= x(n) + 0.25x(n-1) + 0.0625x(n-2) + \ldots \end{aligned}$$

d. From the difference equation and using the zero-initial condition, we have

$$y(n) = 0.25y(n-1) + x(n) \text{ for } n \geq 0 \text{ and } y(-1) = 0$$
$$n = 0, y(0) = 0.25y(-1) + x(0) = u(0) = 1$$
$$n = 1, y(1) = 0.25y(0) + x(1) = 0.25 \times u(0) + u(1) = 1.25$$
$$n = 2, y(2) = 0.25y(1) + x(2) = 0.25 \times 1.25 + u(2) = 1.3125$$
$$\ldots.$$

Applying the convolution sum in Equation (3.15) yields

$$y(n) = x(n) + 0.25x(n-1) + 0.0625x(n-2) + \ldots$$

$$\begin{aligned} n = 0,\ y(0) &= x(0) + 0.25x(-1) + 0.0625x(-2) + \ldots \\ &= u(0) + 0.25 \times u(-1) + 0.125 \times u(-2) + \ldots = 1 \end{aligned}$$

$$\begin{aligned} n = 1,\ y(1) &= x(1) + 0.25x(0) + 0.0625x(-1) + \ldots \\ &= u(1) + 0.25 \times u(0) + 0.125 \times u(-1) + \ldots = 1.25 \end{aligned}$$

$$\begin{aligned} n = 2,\ y(2) &= x(2) + 0.25x(1) + 0.0625x(0) + \ldots \\ &= u(2) + 0.25 \times u(1) + 0.0625 \times u(0) + \ldots = 1.3125 \end{aligned}$$

“... a linear time-invariant system can be represented by the convolution sum using its impulse response and input sequence.”

3.10. Find the unit-impulse response for each of the following linear systems.

a. $y(n) = 0.5x(n) - 0.5x(n-2)$; for $n \geq 0$, $x(-2) = 0$, $x(-1) = 0$

b. $y(n) = 0.75y(n-1) + x(n)$; for $n \geq 0$, $y(-1) = 0$

c. $y(n) = -0.8y(n-1) + x(n-1)$; for $n \geq 0$, $x(-1) = 0$, $y(-1) = 0$

3.11. For each of the following linear systems, find the unit-impulse response, and draw the block diagram.

a. $y(n) = 5x(n-10)$

b. $y(n) = x(n) + 0.5x(n-1)$

Digital convolution sum

A linear time-invariant system can be represented by using a digital convolution sum. Given a linear time-invariant system, we can determine its unit-impulse response h(n), which relates the system input and output. (The sequences h(k) and x(k) in equations are interchangeable).

$$y(n) = h(n) * x(n).$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$= \ldots + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

$$= \ldots + x(-1)h(n+1) + x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) + \ldots$$

... para un sistema causal

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) = \sum_{k=0}^{\infty} x(k)h(n-k).$$

Methods to implement convolution

- Graphical (need reverse and shifted sequences)
- Formula
- Table

The reversed sequence is a mirror image of the original sequence, assuming the vertical axis as the mirror (If h(n) is the given sequence, h(-n) is the reversed sequence)

Example

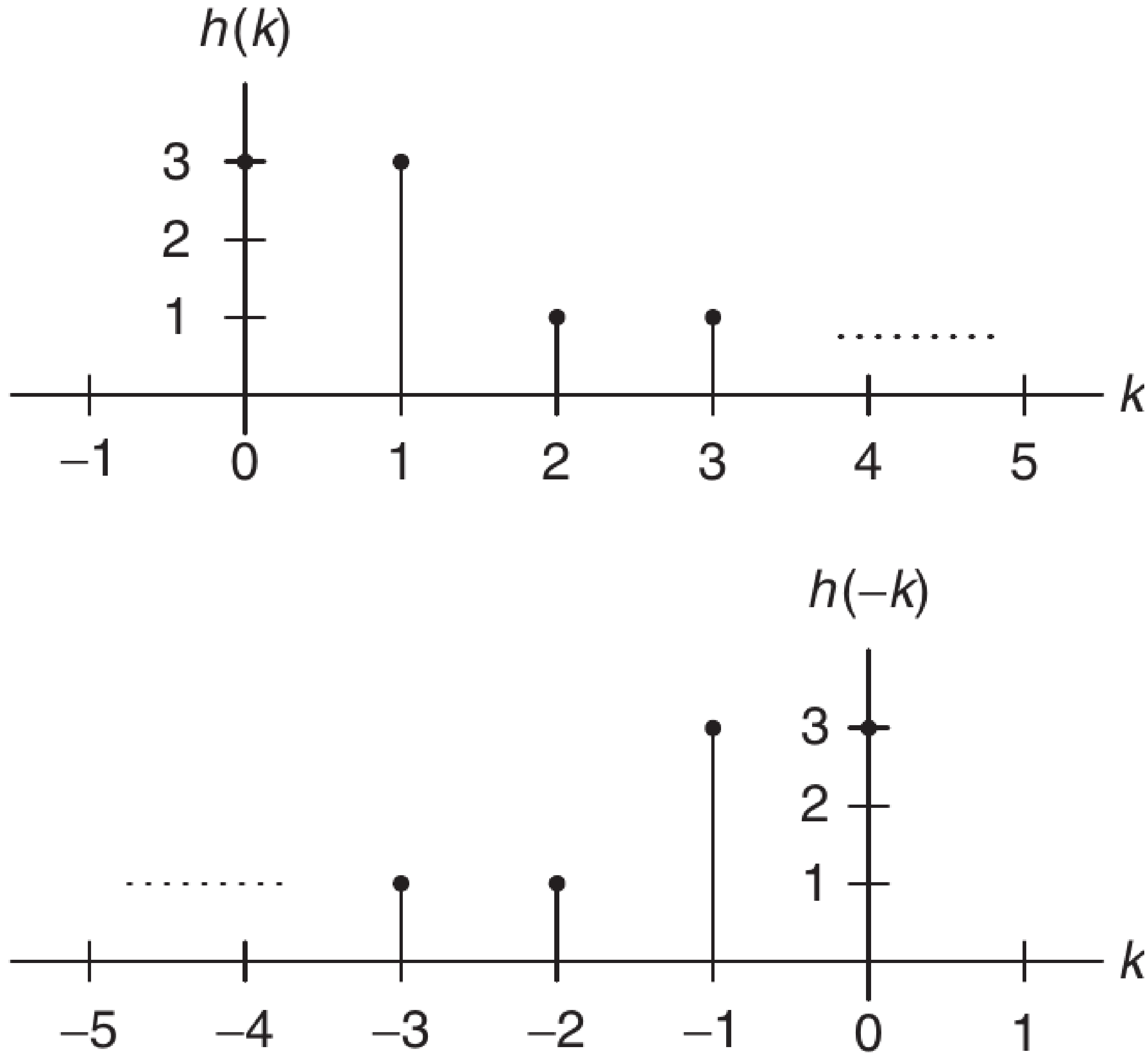
Given a sequence,

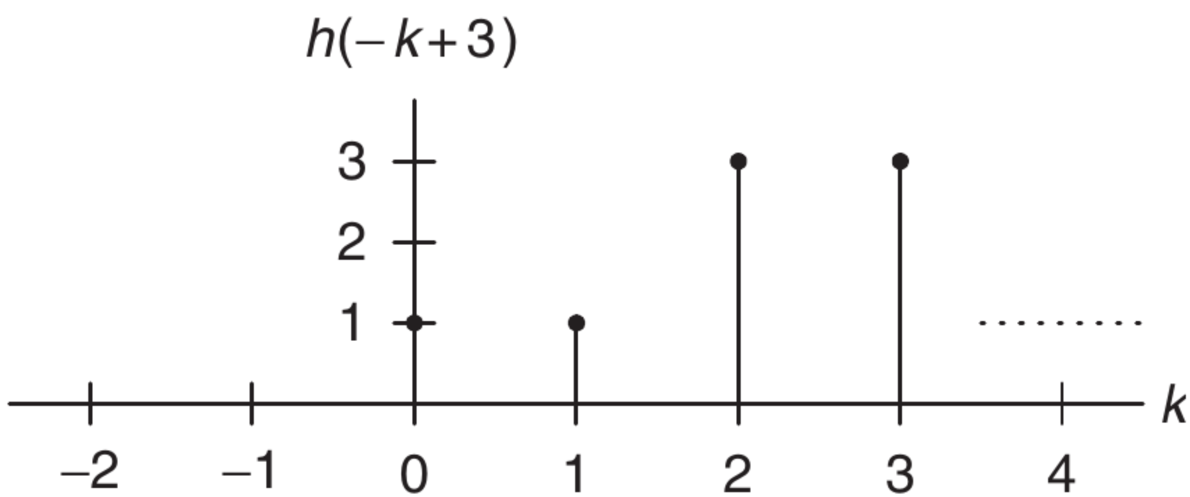
$$h(k) = \begin{cases} 3, & k = 0,1 \\ 1, & k = 2,3 \\ 0 & elsewhere \end{cases}$$

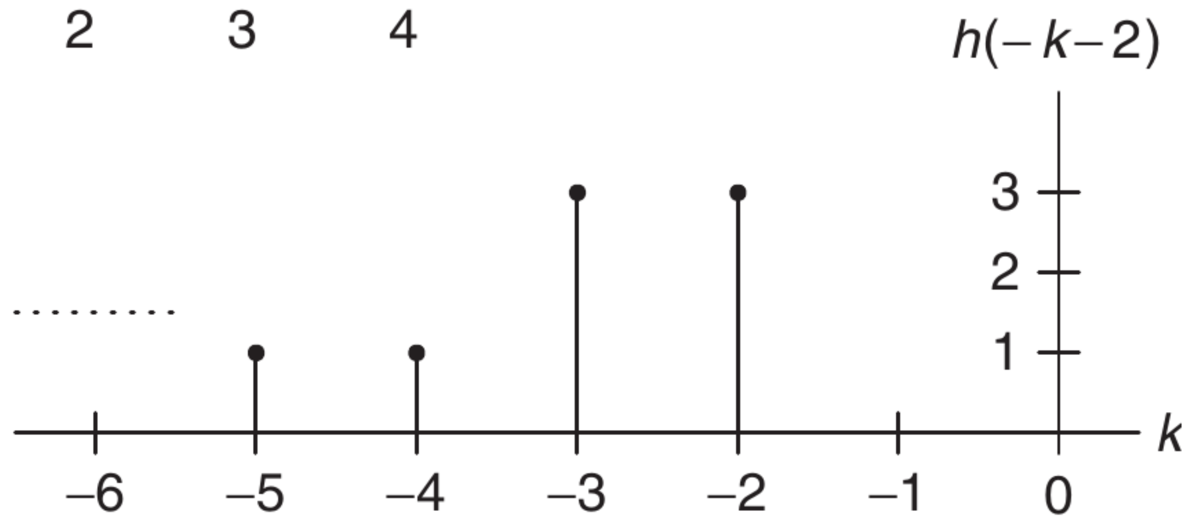
where k is the time index or sample number,

a. Sketch the sequence $h(k)$ and reversed sequence $h(-k)$.

b. Sketch the shifted sequences $h(-k+3)$ and $h(-k-2)$.


h(k)
3
2
1
k
–1
0
1
2
3
4
5
h(–k)
3
2
1
k
–5
–4
–3
–2
–1
0
1

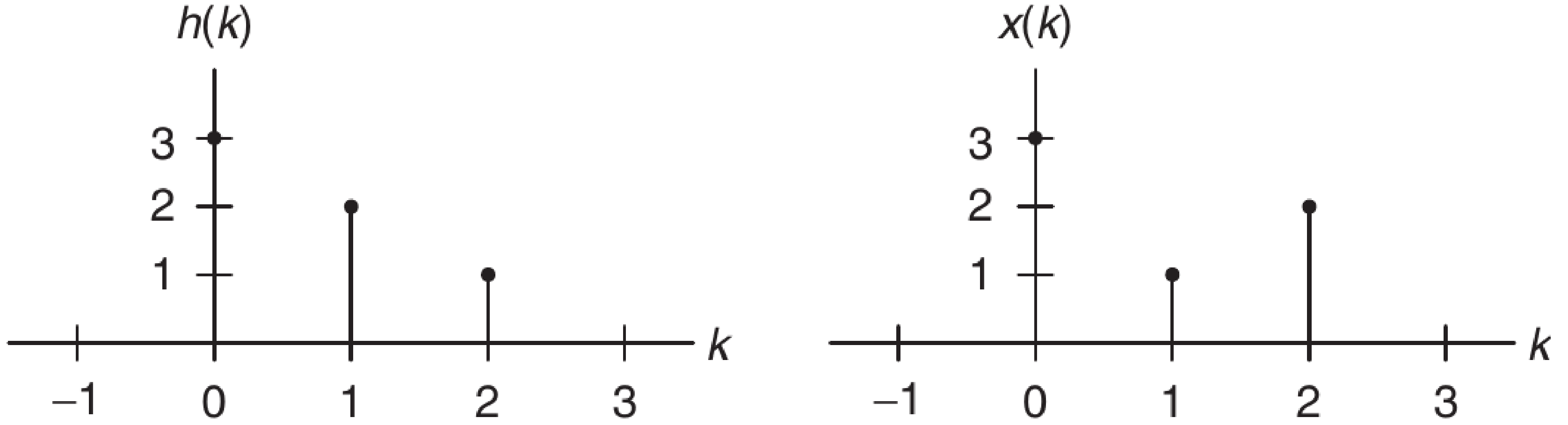

h(−k+3)
3
2
1
−2
−1
0
1
2
3
4
k


h(−k−2)
3
2
1
−6
−5
−4
−3
−2
−1
0
k

Example

Using the following sequences defined in Figure 3.21, evaluate the digital convolution

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

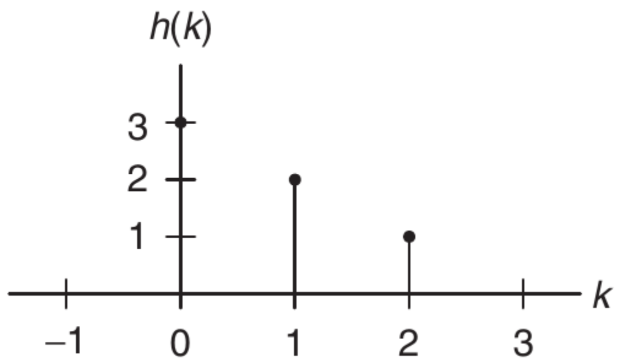


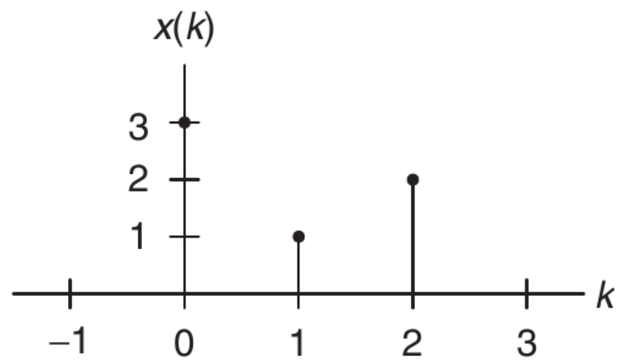
a. By the graphical method.

b. By applying the formula directly.

c. using the table method

Graphical method





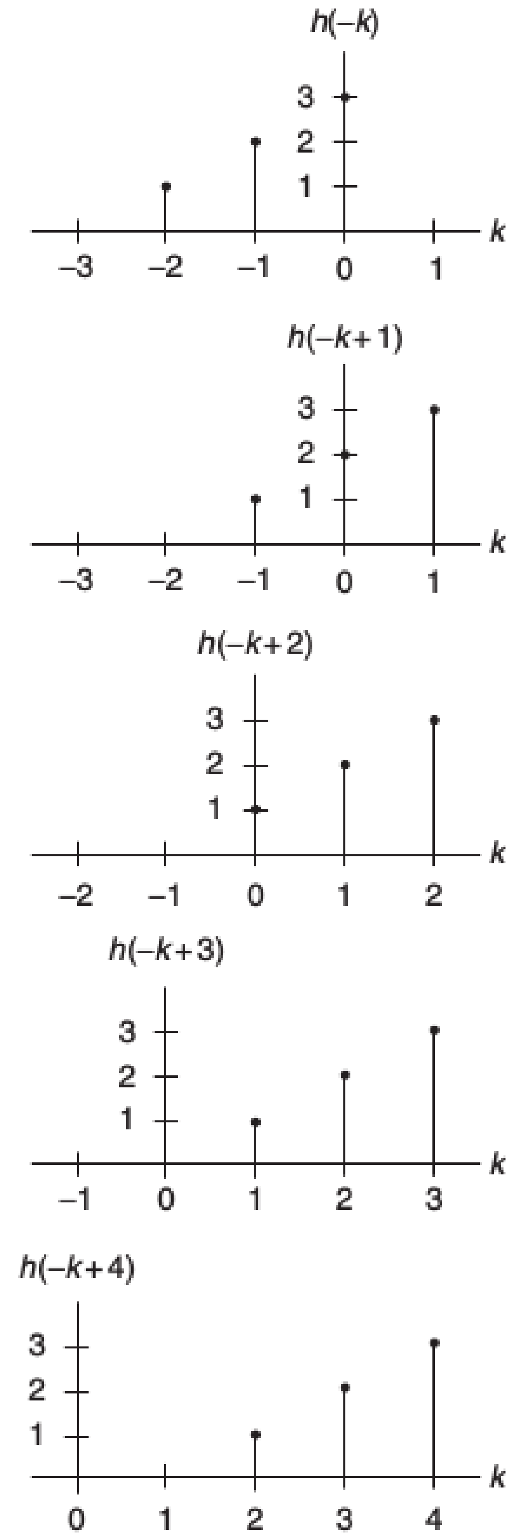
sum of product of $x(k)$ and $h(-k)$: $y(0) = 3 \times 3 = 9$

sum of product of $x(k)$ and $h(1-k)$: $y(1) = 1 \times 3 + 3 \times 2 = 9$

sum of product of $x(k)$ and $h(2-k)$: $y(2) = 2 \times 3 + 1 \times 2 + 3 \times 1 = 11$

sum of product of $x(k)$ and $h(3-k)$: $y(3) = 2 \times 2 + 1 \times 1 = 5$

sum of product of $x(k)$ and $h(4-k)$: $y(4) = 2 \times 1 = 2$

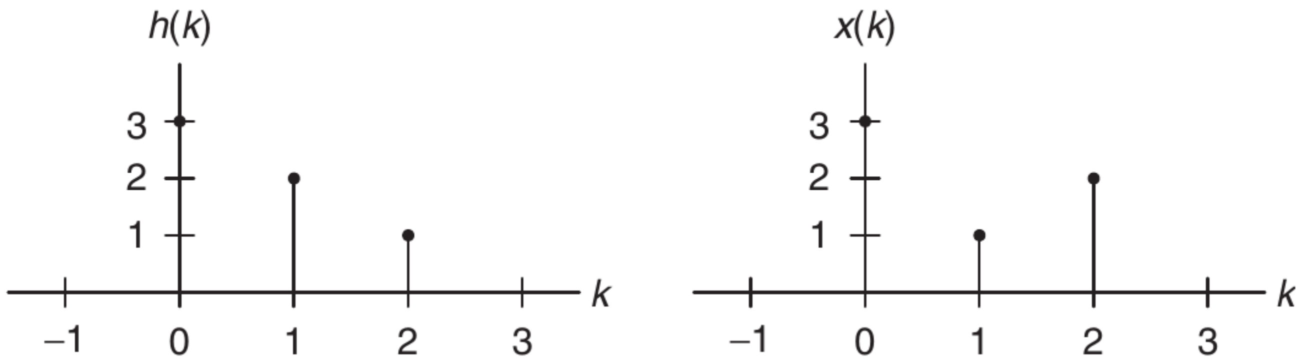


Animation

Animación 1

Animación 2

Formula method

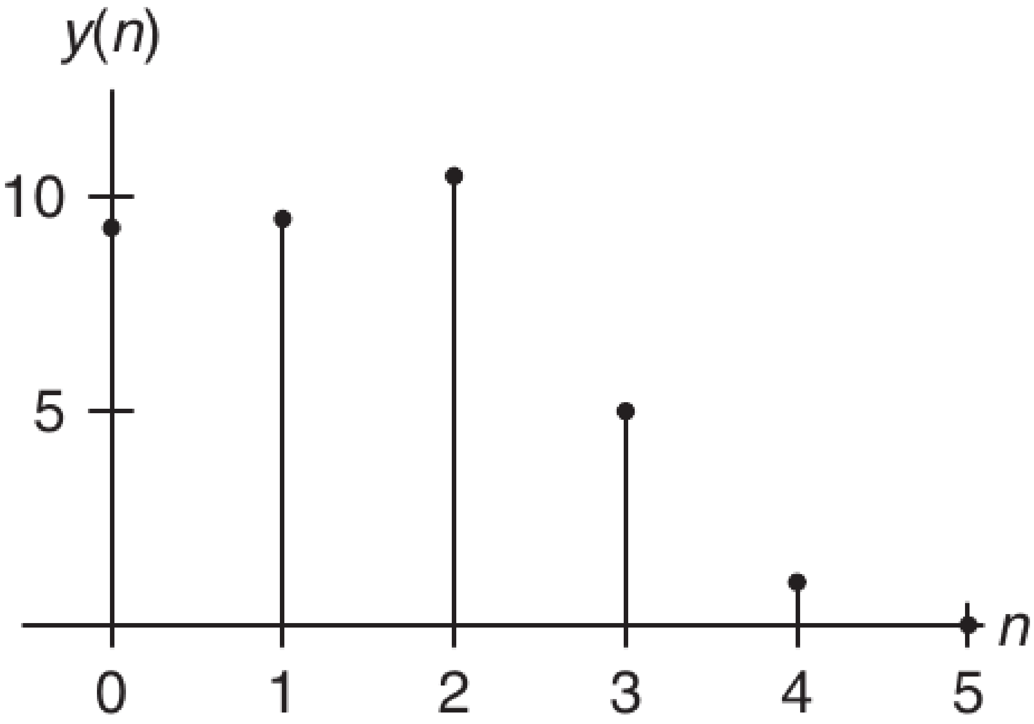


$$n = 0,\ y(0) = x(0)h(0) + x(1)h(-1) + x(2)h(-2) = 3 \times 3 + 1 \times 0 + 2 \times 0 = 9,$$
$$n = 1,\ y(1) = x(0)h(1) + x(1)h(0) + x(2)h(-1) = 3 \times 2 + 1 \times 3 + 2 \times 0 = 9,$$
$$n = 2,\ y(2) = x(0)h(2) + x(1)h(1) + x(2)h(0) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11,$$
$$n = 3,\ y(3) = x(0)h(3) + x(1)h(2) + x(2)h(1) = 3 \times 0 + 1 \times 1 + 2 \times 2 = 5.$$
$$n = 4,\ y(4) = x(0)h(4) + x(1)h(3) + x(2)h(2) = 3 \times 0 + 1 \times 0 + 2 \times 1 = 2,$$
$$n \geq 5,\ y(n) = x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) = 3 \times 0 + 1 \times 0 + 2 \times 0 = 0.$$

Table method

k:	-2	-1	0	1	2	3	4	5	
$x(k)$:			**3**	**1**	**2**				
$h(-k)$:	1	2	3						$y(0) = 3 \times 3 = 9$
$h(1-k)$		1	2	3					$y(1) = 3 \times 2 + 1 \times 3 = 9$
$h(2-k)$			1	2	3				$y(2) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11$
$h(3-k)$				1	2	3			$y(3) = 1 \times 1 + 2 \times 2 = 5$
$h(4-k)$					1	2	3		$y(4) = 2 \times 1 = 2$
$h(5-k)$						1	2	3	$y(5) = 0$ (no overlap)

Convolution



```
>> h=[3 2 1];
>> x=[3 1 2];
>> conv(h,x)

ans =

     9     9    11     5     2
```

Example

A system representation using the unit-impulse response for the linear system

$$y(n) = 0.25y(n-1) + x(n) \text{ for } n \geq 0 \text{ and } y(-1) = 0$$

is determined in Example 3.8 as

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k),$$

where $h(n) = (0.25)^n u(n)$. For a step input $x(n) = u(n)$,

a. Determine the output response for the first three output samples using the table method.

k:	-2	-1	0	1	2	3	…	
$x(k)$:			**1**	**1**	**1**	**1**	…	
$h(-k)$:	0.0625	0.25	1					$y(0) = 1 \times 1 = 1$
$h(1-k)$		0.0625	0.25	1				$y(1) = 1 \times 0.25 + 1 \times 1 = 1.25$
$h(2-k)$			0.0625	0.25	1			$y(2) = 1 \times 0.0625 + 1 \times 0.25 + 1 \times 1$ $= 1.3125$
								Stop as required

Problems

3.15. Using the following sequence definitions,

$$h(k) = \begin{cases} 2, & k = 0,1,2 \\ 1, & k = 3,4 \\ 0 & elsewhere \end{cases} \quad \text{and} \quad x(k) = \begin{cases} 2, & k = 0 \\ 1, & k = 1,2 \\ 0 & elsewhere, \end{cases}$$

evaluate the digital convolution

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

a. using the graphical method;

b. using the table method;

c. applying the convolution formula directly.

3.16. Using the sequence definitions

$$x(k) = \begin{cases} -2, & k = 0,1,2 \\ 1, & k = 3,4 \\ 0 & elsewhere \end{cases} \quad \text{and} \quad h(k) = \begin{cases} 2, & k = 0 \\ -1, & k = 1,2 \\ 0 & elsewhere, \end{cases}$$

evaluate the digital convolution

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

a. using the graphical method;

b. using the table method;

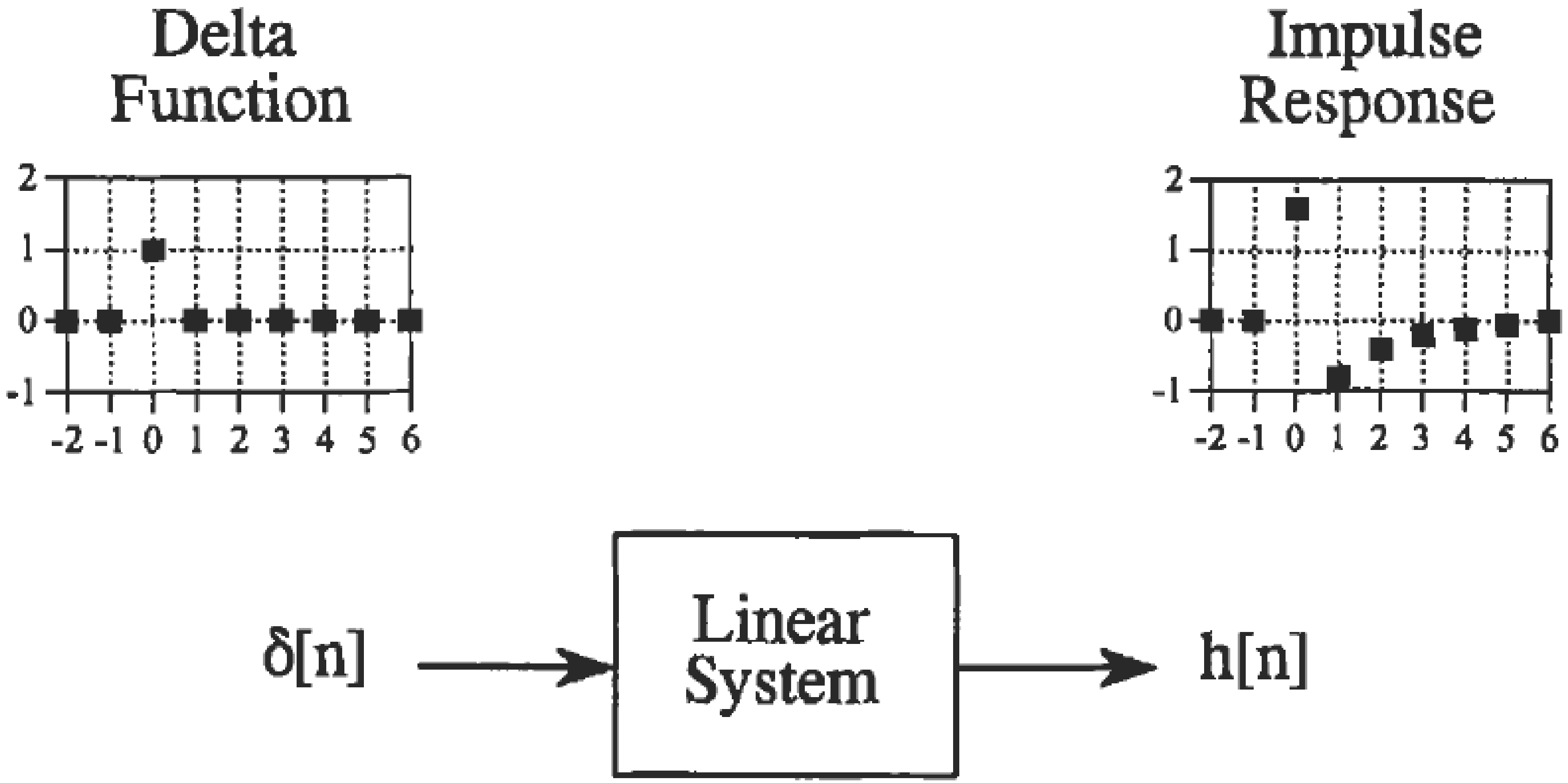
c. applying the convolution formula directly.

3.17. Convolve the following two rectangular sequences:

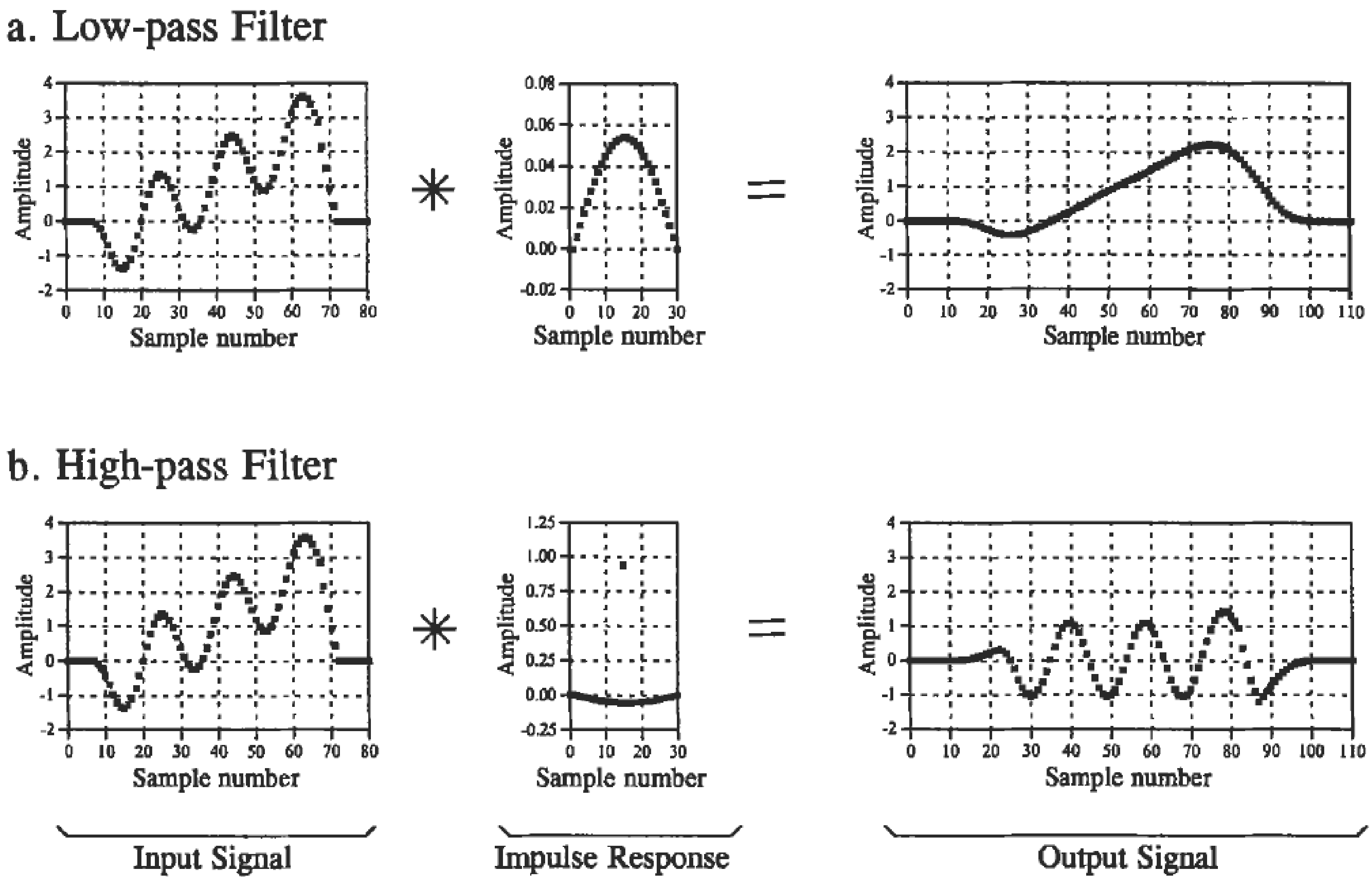
$$x(n) = \begin{cases} 1 & n = 0,1 \\ 0 & otherwise \end{cases} \quad \text{and} \quad h(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1,2 \\ 0 & otherwise \end{cases}$$

using the table method.

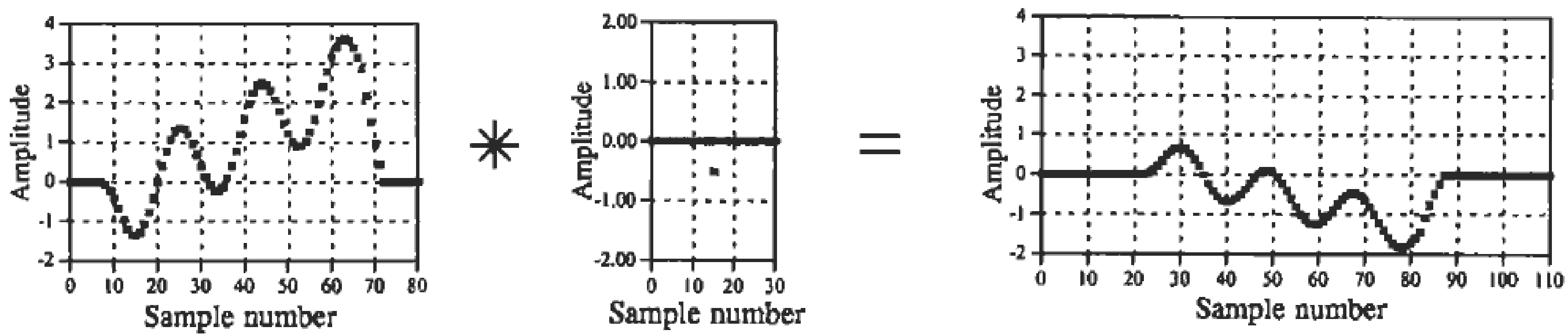
Convolution



Examples convolution



a. Inverting Attenuator



b. Discrete Derivative