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Complex Analysis and Operator Theory

Singular Integral Operators with Coefficients of a Special Structure Related to Operator Equalities

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Abstract. In our previous works we have constructed operator equalities which transform scalar singular integral operators with shift to matrix characteristic singular integral operators without shift and found some of their applications to problems with shift. In this article the operator equalities are used for the study of matrix characteristic singular integral operators.

Conditions for the invertibility of the singular integral operators with orientation preserving shift and coefficients with a special structure generated by piecewise constant functions, t, t^{-1}, were found.

Conditions for the invertibility of the matrix characteristic singular integral operators with four-valued piecewise constant coefficients of a special structure were likewise obtained.

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1. Introduction

We denote by $[B_1, B_2]$ the set of all bounded linear operators mapping the Banach space B_1 into the Banach space B_2, $[B_1] \equiv [B_1, B_1]$. It is known [1, 2] that for any operator $A = X + ZY$, where $X, Y, Z \in [B_1]$ and Z is an involutive operator, $Z^2 = I$, the Gohberg–Krupnik matrix equality is fulfilled:

$$H \begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix} H^{-1} = \mathcal{D}\,,$$

where A_1 is an additional associated operator, $A_1 = X - ZY$, and

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ Z & -Z \end{bmatrix}, \quad H^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} I & Z \\ I & -Z \end{bmatrix}, \quad \mathcal{D} = \begin{bmatrix} X & ZYZ \\ Y & ZXZ \end{bmatrix}.$$

We denote the Cauchy singular integral operator along a contour Γ by

$$(S_\Gamma\varphi)(t) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau$$

and the identity operator on Γ by $(I_\Gamma\varphi)(t) = \varphi(t)$.

Suppose that

$$X = aI_\Gamma + cS_\Gamma\,, \quad Y = (Zb)I_\Gamma + (Zd)S_\Gamma\,,$$

where a, b, c, d are bounded measurable functions on Γ and $(Z_\Gamma\varphi)(\tau) = \varphi(-\tau)$. We denote the unit circle by $\mathbb{T}$ and the real axis by $\mathbb{R}$. The matrix equality takes the form

$$H\begin{bmatrix} aI_\Gamma + bZ_\Gamma + cS_\Gamma + dZ_\Gamma S_\Gamma & 0 \\ 0 & aI_\Gamma - bZ_\Gamma + cS_\Gamma - dZ_\Gamma S_\Gamma \end{bmatrix}H^{-1} = \mathcal{D}_\Gamma\,, \quad (1.1)$$

where $\mathcal{D}_\Gamma$ is a matrix characteristic singular integral operator:

$$\mathcal{D}_\mathbb{T} = \begin{bmatrix} a & b \\ (Z_\mathbb{T}b) & (Z_\mathbb{T}a) \end{bmatrix} I_\mathbb{T} + \begin{bmatrix} c & d \\ (Z_\mathbb{T}d) & (Z_\mathbb{T}c) \end{bmatrix} S_\mathbb{T}\,,$$

$$\mathcal{D}_\mathbb{R} = \begin{bmatrix} a & b \\ (Z_\mathbb{R}b) & (Z_\mathbb{R}a) \end{bmatrix} I_\mathbb{R} + \begin{bmatrix} c & -d \\ (Z_\mathbb{R}d) & (-Z_\mathbb{R}c) \end{bmatrix} S_\mathbb{R}\,.$$

The operators $\mathcal{D}_\mathbb{T}$ and $\mathcal{D}_\mathbb{R}$ are different because Z is an orientation-preserving shift on $\mathbb{T}$, $Z_\mathbb{T}S_\mathbb{T} = S_\mathbb{T}Z_\mathbb{T}$, but on $\mathbb{R}$ it is an orientation-reversing shift , $Z_\mathbb{R}S_\mathbb{R} = -S_\mathbb{R}Z_\mathbb{R}$.

In the article [3] we obtained a direct relation between the operator A with a model involutive operator and a matrix characteristic singular integral operator without additional associated operators: for an orientation-preserving shift it is a similarity transform $\mathcal{F}A\mathcal{F}^{-1}$, and for an orientation-reversing shift it is a transform by two invertible operators $\mathcal{H}A\mathcal{E}$. We formulate these results below in Section 2.

In this paper, the Gohberg–Krupnik matrix equality (1.1) and the transforms from [3] are the main tools for the study of questions connected with the invertibility problem.

In Section 3, we consider a singular integral operator with orientation preserving shift on the unit circle and coefficients generated by piecewise constant functions, functions t and t^{-1}, and possessing special properties. Using the operator equality corresponding of the orientation-preserving shift we obtain conditions for the invertibility of the operator.

In Section 4, we consider a characteristic matrix singular integral operator. The coefficients of the operator are piecewise constant matrix functions with, at most, four different values and possessing special properties. Using the operator equality corresponding of the orientation-reversing shift we obtain conditions for the invertibility of the operator.

2. Operator equalities

Let Γ and γ be contours, and $\gamma \subset \Gamma$. The extension of a function $f(t), t \in \gamma$, to $\Gamma \setminus \gamma$ by the value zero, will be denoted by $(J_{\Gamma\setminus\gamma}f)(t), t \in \Gamma$. The restriction of a function $\varphi(t), t \in \Gamma$, to γ will be denoted by $(C_\gamma\varphi)(t)$, $t \in \gamma$. The characteristic function of the set γ given on Γ will be denoted by $\chi_\gamma(t)$, $t \in \Gamma$.

Let $L_p(\Gamma, \rho)$ denote the space of functions on Γ which are summable in the p-th power after multiplication by the weight-function ρ, and let $L_p^m(\Gamma, \rho)$ denote the space of m-dimensional vector-functions with components from $L_p(\Gamma, \rho)$.

We define

$$\mathcal{L} = \{z : |z| = 1,\ 0 < \arg z < 2\pi/m\}\,;$$
$$(W_m\varphi)(t) = \varphi(\varepsilon_m t)\,, \quad \varepsilon_m := \cos\frac{2\pi}{m} + i\sin\frac{2\pi}{m}$$

and

$$M\begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_m \end{bmatrix} = \sum_{k=1}^{m} W_m^{-k+1} J_{\mathbb{T}\setminus\mathcal{L}}\varphi_k\,, \quad M \in \big[L_2^m(\mathcal{L}), L_2(\mathbb{T})\big]\,,$$
$$M^{-1}\varphi = \begin{bmatrix} C_{\mathcal{L}}\varphi \\ C_{\mathcal{L}}W_m\varphi \\ \vdots \\ C_{\mathcal{L}}W_m^{m-1}\varphi \end{bmatrix};$$
$$\Pi = \frac{1}{\sqrt{m}}\big[\varepsilon^{(r-1)(k-r-1)}\big]_{k,r=1}^{m}\,, \quad \Pi^{-1} = \frac{1}{\sqrt{m}}\big[\varepsilon^{(k-1)(k-r+1)}\big]_{k,r=1}^{m}\,;$$
$$V = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & & \ddots & 1 \\ 1 & & & & 0 \end{bmatrix},$$
$$\Pi^{-1}V\Pi = \Omega\,, \quad \Omega = \operatorname{diag}\,[1, \varepsilon^1, \dots, \varepsilon^{m-1}]\,;$$
$$G(t) = \operatorname{diag}\,[1, t^1, \dots, t^{m-1}]\,, \quad G^{-1}(t) = \operatorname{diag}\,[1, t^{-1}, \dots, t^{1-m}]\,, \quad t \in \mathcal{L}\,;$$
$$(N\zeta)(t) = \zeta(t^m)\,, \quad N \in \big[L_2^m(\mathbb{T}), L_2^m(\mathcal{L})\big]\,,$$
$$(N^{-1}\zeta)(t) = \zeta(t^{\frac{1}{m}})\,.$$

Theorem 2.1 ([3], Theorem 2.16, p. 240). *The singular integral operator A with the shift-rotation W_m at the angle $2\pi/m$ and bounded measurable coefficients,*

$$A = \sum_{k=0}^{m-1}\big[a_k(t)I_{\mathbb{T}} + b_k(t)S_{\mathbb{T}}\big]W_m^k\,, \quad A \in \big[L_2(\mathbb{T})\big]\,,$$

is similar to the matrix characteristic singular integral operator $D_{\mathbb{T}}$:

$$D_{\mathbb{T}} = \mathcal{F}^{-1}A\mathcal{F}\,, \quad D_{\mathbb{T}} = uI_{\mathbb{T}} + vS_{\mathbb{T}}\,, \quad D_{\mathbb{T}} \in \big[L_2^m(\mathbb{T})\big]\,, \tag{2.1}$$

where

$$\mathcal{F} = M\Pi GN \in \big[L_2^m(\mathbb{T}), L_2(\mathbb{T})\big]\,, \quad \mathcal{F}^{-1} = N^{-1}G^{-1}\Pi^{-1}M^{-1} \in \big[L_2(\mathbb{T}), L_2^m(\mathbb{T})\big]\,.$$

The connection between the coefficients of the operator A and the coefficients of the operator $D_{\mathbb{T}}$ is given by the formulas:

$$\begin{aligned} u(t) &= \left[\frac{t^{(1-k)/m}\varepsilon^{(k-1)(k-r+1)}}{\sqrt{m}}\right]_{k,r=1}^{m} u_1(t^{1/m}) \left[\frac{\varepsilon^{(r-1)(k-r-1)}t^{(r-1)/m}}{\sqrt{m}}\right]_{k,r=1}^{m}, \\ v(t) &= \left[\frac{t^{(1-k)/m}\varepsilon^{(k-1)(k-r+1)}}{\sqrt{m}}\right]_{k,r=1}^{m} v_1(t^{1/m}) \left[\frac{\varepsilon^{(r-1)(k-r-1)}t^{(r-1)/m}}{\sqrt{m}}\right]_{k,r=1}^{m}, \quad t\in\mathbb{T}, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} u_1(t) &= \begin{bmatrix} a_0(t) & a_1(t) & \dots & a_{m-1}(t) \\ a_{m-1}(\varepsilon t) & a_0(\varepsilon t) & \dots & a_1(\varepsilon t) \\ \vdots & \vdots & \ddots & \vdots \\ a_1(\varepsilon^{m-1}t) & a_2(\varepsilon^{m-1}t) & \dots & a_0(\varepsilon^{m-1}t) \end{bmatrix}, \\ v_1(t) &= \begin{bmatrix} b_0(t) & b_1(t) & \dots & b_{m-1}(t) \\ b_{m-1}(\varepsilon t) & b_0(\varepsilon t) & \dots & b_1(\varepsilon t) \\ \vdots & \vdots & \ddots & \vdots \\ b_1(\varepsilon^{m-1}t) & b_2(\varepsilon^{m-1}t) & \dots & b_0(\varepsilon^{m-1}t) \end{bmatrix}, \quad t \in \mathcal{L}. \end{aligned} \tag{2.3}$$

We now formulate a theorem for the case of an orientation reversing shift. We denote the positive semi-axis by $\mathbb{R}_+ = (0, +\infty)$ and the negative semi-axis by $\mathbb{R}_- = (-\infty, 0)$;

$$\begin{aligned} (Q\varphi)(x) &= \frac{\sqrt{\delta^2+\beta}}{x-\delta}\varphi\big[\alpha(x)\big], \\ \alpha(x) &= \frac{\delta x+\beta}{x-\delta}, \quad \delta\in\mathbb{R}, \quad \beta\in\mathbb{R}, \quad \delta^2+\beta>0, \end{aligned} \tag{2.4}$$

the operator Q is generated by a Carleman linear-fractional orientation-reversing shift, $\alpha(\alpha(x)) \equiv x$ and has a property $Q^2 = I_{\mathbb{R}}$;

$$(\Theta\varphi)(x) = \frac{x_2-x_1}{x_2-x}\varphi\left(\frac{x-x_1}{x_2-x}\right), \quad (\Theta^{-1}\varphi)(x) = \frac{1}{x+1}\varphi\left(\frac{x_2x+x_1}{x+1}\right),$$

where

$$\begin{aligned} x_1 &= \delta - \sqrt{\delta^2+\beta}, & x_2 &= \delta + \sqrt{\delta^2+\beta}; \\ \big(N_{\mathbb{R}_+}\varphi\big)(t) &= \varphi(t^2), & \big(N_{\mathbb{R}_+}^{-1}\varphi\big)(t) &= \varphi(\sqrt{t}); \\ P &= \begin{bmatrix} S_{\mathbb{R}_+}+U_{1,\mathbb{R}_+} & 0 \\ 0 & I_{\mathbb{R}_+} \end{bmatrix}; & \Pi^{\pm1} &= \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}; \\ M_{\mathbb{R}_+}\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} &= \begin{cases} \varphi_1(t), & t\in\mathbb{R}_+ \\ \varphi_2(-t), & t\in\mathbb{R}_- \end{cases}, & M_{\mathbb{R}_+}^{-1}\varphi &= \begin{bmatrix} \varphi(t) \\ \varphi(-t) \end{bmatrix}, \qquad t\in\mathbb{R}_+; \end{aligned}$$

$$\Theta \in \big[L_2(\mathbb{R})\big],\ P \in \big[L_2^2(\mathbb{R}_+)\big],\ N_{\mathbb{R}_+} \in \big[L_2^2(\mathbb{R}_+, t^{-\frac{1}{4}}), L_2^2(\mathbb{R}_+)\big]\,, \\ M_{\mathbb{R}_+} \in \big[L_2^2(\mathbb{R}_+), L_2(\mathbb{R})\big]\,.$$

Theorem 2.2 ([3], Theorem 3.11, p. 244). *The singular integral operator B with bounded measurable coefficients,*

$$B = aI_{\mathbb{R}} + bQ + cS_{\mathbb{R}} + dQS_{\mathbb{R}}\,, \quad B \in [L_2(\mathbb{R})]\,,$$

can be reduced by invertible operators to the matrix characteristic singular integral operator $D_{\mathbb{R}_+}$:

$$D_{\mathbb{R}_+} = \mathcal{H}B\mathcal{E}\,, \quad D_{\mathbb{R}_+} = uI_{\mathbb{R}_+} + vS_{\mathbb{R}_+}\,, \quad D_{\mathbb{R}_+} \in \left[L_2^2\left(\mathbb{R}_+, t^{-\frac{1}{4}}\right)\right], \tag{2.5}$$

where

$$\mathcal{H} = N_{\mathbb{R}_+}^{-1}\Pi^{-1}M_{\mathbb{R}}^{-1}\Theta^{-1}\,, \qquad \mathcal{H} \in \left[L_2^2(\mathbb{R}_+), L_2^2\left(\mathbb{R}_+, t^{-\frac{1}{4}}\right)\right],$$

$$\mathcal{E} = \Theta M_{\mathbb{R}}\Pi P N_{\mathbb{R}_+}\,, \qquad \mathcal{E} \in \left[L_2^2\left(\mathbb{R}_+, t^{-\frac{1}{4}}\right), L_2^2(\mathbb{R}_+)\right].$$

The relation between the coefficients of the operator B and the coefficients of the operator $D_{\mathbb{R}_+}$ is given by the formulas:

$$u(t) = \frac{1}{2}\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}, \quad v(t) = \frac{1}{2}\begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}, \tag{2.6}$$

where

$$\begin{aligned}
u_{11}(t) &= \Big(c\big(\zeta_+(t)\big) + d\big(\zeta_+(t)\big)\Big) - \Big(c\big(\zeta_-(t)\big) + d\big(\zeta_-(t)\big)\Big)\,,\\
u_{12}(t) &= \Big(a\big(\zeta_+(t)\big) + b\big(\zeta_+(t)\big)\Big) - \Big(a\big(\zeta_-(t)\big) + b\big(\zeta_-(t)\big)\Big)\,,\\
u_{21}(t) &= \Big(c\big(\zeta_+(t)\big) + d\big(\zeta_+(t)\big)\Big) + \Big(c\big(\zeta_-(t)\big) + d\big(\zeta_-(t)\big)\Big)\,,\\
u_{22}(t) &= \Big(a\big(\zeta_+(t)\big) + b\big(\zeta_+(t)\big)\Big) + \Big(a\big(\zeta_-(t)\big) + b\big(\zeta_-(t)\big)\Big)\,,\\
v_{11}(t) &= \Big(a\big(\zeta_+(t)\big) - b\big(\zeta_+(t)\big)\Big) + \Big(a\big(\zeta_-(t)\big) - b\big(\zeta_-(t)\big)\Big)\,,\\
v_{12}(t) &= \Big(c\big(\zeta_+(t)\big) - d\big(\zeta_+(t)\big)\Big) + \Big(c\big(\zeta_-(t)\big) - d\big(\zeta_-(t)\big)\Big)\,,\\
v_{21}(t) &= \Big(a\big(\zeta_+(t)\big) - b\big(\zeta_+(t)\big)\Big) - \Big(a\big(\zeta_-(t)\big) - b\big(\zeta_-(t)\big)\Big)\,,\\
v_{22}(t) &= \Big(c\big(\zeta_+(t)\big) - d\big(\zeta_+(t)\big)\Big) - \Big(c\big(\zeta_-(t)\big) - d\big(\zeta_-(t)\big)\Big)\,, \qquad (2.7)\\
\zeta_+(t) &= \frac{x_2\sqrt{t} + x_1}{\sqrt{t}+1}\,, \quad \zeta_-(t) = \frac{-x_2\sqrt{t} + x_1}{-\sqrt{t}+1}\,, \quad t \in \mathbb{R}_+\,.
\end{aligned}$$

We will refer to formulas (2.1) and (2.5) as operator equalities. The operator equalities will be used to study the invertibility properties of singular integral operators.

3. Singular integral operators with coefficients of a special structure generated by piecewise constant functions, t and t^{-1}

We denote the upper semicircle of $\mathbb{T}$ by $\mathbb{T}_+$ and the lower semicircle of $\mathbb{T}$ by $\mathbb{T}_-$.

Let $a_{2,ij}$ and $b_{2,ij}$, where $i = 1, 2,\ \ j = 1, 2$ be piecewise constant functions given on $\mathbb{T}_+$, with three values and points of discontinuity at $t = t_0,\ t = t_1$, $0 < arg t_0 < arg t_1 < \pi$.

In the space $L_2(\mathbb{T})$, let us consider the operator

$$\begin{gathered} A_{\mathbb{T}} = a_{\mathbb{T}} I_{\mathbb{T}} + c_{\mathbb{T}} S_{\mathbb{T}} + b_{\mathbb{T}} W_{\mathbb{T}} + d_{\mathbb{T}} S_{\mathbb{T}} W_{\mathbb{T}}\,, \\ (W_{\mathbb{T}}\varphi)(t) = \varphi(-t)\,, \quad A_{\mathbb{T}} \in \big[L_2(\mathbb{T})\big]\,, \end{gathered} \tag{3.1}$$

with coefficients generated by $a_{2,ij},\ b_{2,ij}$, and functions $t,\ t^{-1}$, and which have the following special properties:

$$\begin{aligned} C_{\mathbb{T}_+} a_{\mathbb{T}} &= \frac{1}{2}\big[a_{2,11} + a_{2,22} + t a_{2,21} + t^{-1} a_{2,12}\big]\,, \\ C_{\mathbb{T}_+} (W_{\mathbb{T}} a_{\mathbb{T}}) &= \frac{1}{2}\big[a_{2,11} + a_{2,22} - t a_{2,21} - t^{-1} a_{2,12}\big]\,, \\ C_{\mathbb{T}_+} b_{\mathbb{T}} &= \frac{1}{2}\big[a_{2,11} - a_{2,22} + t a_{2,21} - t^{-1} a_{2,12}\big]\,, \\ C_{\mathbb{T}_+} (W_{\mathbb{T}} b_{\mathbb{T}}) &= \frac{1}{2}\big[a_{2,11} - a_{2,22} - t a_{2,21} + t^{-1} a_{2,12}\big]\,, \\ C_{\mathbb{T}_+} c_{\mathbb{T}} &= \frac{1}{2}\big[b_{2,11} + b_{2,22} + t b_{2,21} + t^{-1} b_{2,12}\big]\,, \\ C_{\mathbb{T}_+} (W_{\mathbb{T}} c_{\mathbb{T}}) &= \frac{1}{2}\big[b_{2,11} + b_{2,22} - t b_{2,21} - t^{-1} b_{2,12}\big]\,, \\ C_{\mathbb{T}_+} d_{\mathbb{T}} &= \frac{1}{2}\big[b_{2,11} - b_{2,22} + t b_{2,21} - t^{-1} b_{2,12}\big]\,, \\ C_{\mathbb{T}_+} (W_{\mathbb{T}} d_{\mathbb{T}}) &= \frac{1}{2}\big[b_{2,11} - b_{2,22} - t b_{2,21} + t^{-1} b_{2,12}\big]\,. \end{aligned} \tag{3.2}$$

Operator equalities have distinctions between the cases where the orientation is preserved and the cases where the orientation is changed.

After applying the operator equality (2.5) the weight $t^{-\frac{1}{4}}$ appears; the coefficients (2.6), (2.7) conserve the property of being piecewise constant (Theorem 2.2).

After applying the operator equality (2.1) a weight does not appear, the coefficients (2.2), (2.3) do not conserve the property of being piecewise constant (Theorem 2.1). The multipliers $t^{r-1/m}$ appear. For $m = 2$, we have $t^{1/2}$.

The coefficients (3.2) of the operator $A_{\mathbb{T}}$ are selected so that the transformed operator $\mathcal{F}^{-1} A_{\mathbb{T}} \mathcal{F}$ has piecewise constant coefficients and we can use the results from [4]; that is, so that the multipliers $t,\ t^{-1}$ of the coefficients (3.2) eliminate the multipliers which appear after applying the operator equality (2.1).

For this reason the coefficients of the operator $B_{\mathbb{R}}$ from Section 4 look more "natural" than the coefficients of the operator $A_{\mathbb{T}}$.

Lemma 3.1. *The operator*

$$A_{\mathbb{T}} = a_{\mathbb{T}} I_{\mathbb{T}} + c_{\mathbb{T}} S_{\mathbb{T}} + b_{\mathbb{T}} W_{\mathbb{T}} + d_{\mathbb{T}} S_{\mathbb{T}} W_{\mathbb{T}} \in \left[L_2(\mathbb{T})\right]$$

with orientation-preserving shift on the unit circle $(W\varphi)(t) = \varphi(-t)$ *and coefficients* (3.2) *is reduced to the matrix characteristic singular integral operator*

$$D_{\mathbb{T}} = u_{\mathbb{T}} I_{\mathbb{T}} + v_{\mathbb{T}} S_{\mathbb{T}} \in \left[L_2^2(\mathbb{T})\right].$$

The coefficients are piecewise constant matrices which are defined on the unit circle and have three values

$$\begin{aligned} u_{\mathbb{T}}(t) &= A_0 \chi_{(0,t_0^2)} + A_1 \chi_{(t_0^2,t_1^2)} + A_2 \chi_{(t_1^2,\pi)} , \\ v_{\mathbb{T}}(t) &= B_0 \chi_{(0,t_0^2)} + B_1 \chi_{(t_0^2,t_1^2)} + B_2 \chi_{(t_1^2,\pi)} , \end{aligned} \tag{3.3}$$

where the constant matrices $A_0, A_1, A_2, B_0, B_1, B_2$ *are*

$$\begin{aligned} A_0 &= C_{(0,t_0)} \begin{bmatrix} a_{2,11} & a_{2,12} \\ a_{2,21} & a_{2,22} \end{bmatrix}, \quad A_1 = C_{(t_0,t_1)} \begin{bmatrix} a_{2,11} & a_{2,12} \\ a_{2,21} & a_{2,22} \end{bmatrix}, \\ A_2 &= C_{(t_1,\pi)} \begin{bmatrix} a_{2,11} & a_{2,12} \\ a_{2,21} & a_{2,22} \end{bmatrix}, \quad B_0 = C_{(0,t_0)} \begin{bmatrix} b_{2,11} & b_{2,12} \\ b_{2,21} & b_{2,22} \end{bmatrix}, \\ B_1 &= C_{(t_0,t_1)} \begin{bmatrix} b_{2,11} & b_{2,12} \\ b_{2,21} & b_{2,22} \end{bmatrix}, \quad B_2 = C_{(t_1,\pi)} \begin{bmatrix} b_{2,11} & b_{2,12} \\ b_{2,21} & b_{2,22} \end{bmatrix}. \end{aligned} \tag{3.4}$$

Proof. We apply the operator equality (2.1) to $A_{\mathbb{T}}$. As a result $A_{\mathbb{T}}$ from (3.1) transforms to the matrix characteristic singular integral operator without shift

$$\mathcal{F}^{-1} A_{\mathbb{T}} \mathcal{F} = D_{\mathbb{T}} , \quad D_{\mathbb{T}} = u_{\mathbb{T}} I_{\mathbb{T}} + v_{\mathbb{T}} S_{\mathbb{T}} \in \left[L_2^2(\mathbb{T})\right]. \tag{3.5}$$

Operator $\mathcal{F} \in [L_2^2(\mathbb{T}), L_2(\mathbb{T})]$ is defined by the composition of the operators: $\mathcal{F} = M\Pi G N$. In our case $m = 2$ and these operators have the following form

$$\begin{aligned} M \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} &= M_{\mathbb{T}_+} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = J_{\mathbb{T}_-} \varphi_1 + W_{\mathbb{T}} J_{\mathbb{T}_-} \varphi_2 , \\ M^{-1}\varphi &= M_{\mathbb{T}_+}^{-1} \varphi = \begin{bmatrix} C_{\mathbb{T}_+} \varphi \\ C_{\mathbb{T}_+} W_{\mathbb{T}} \varphi \end{bmatrix}, \\ \Pi^{\pm 1} &= Z^{\pm 1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \\ G^{\pm 1}(t) &= G_{\mathbb{T}_+}^{\pm 1}(t) = \operatorname{diag}(1, t^{\pm 1}) , \\ (N_{\mathbb{T}_+}\zeta)(t) &= \zeta(t^2) , \\ (N_{\mathbb{T}_+}^{-1}\zeta)(t) &= \zeta(t^{\frac{1}{2}}) , \\ M_{\mathbb{T}_+}^{-1} &\in [L_2(\mathbb{T}), L_2^2(\mathbb{T}_+)] , \\ M_{\mathbb{T}_+} &\in [L_2^2(\mathbb{T}_+), L_2(\mathbb{T})] , \\ N_{\mathbb{T}_+} &\in [L_2^2(\mathbb{T}), L_2^2(\mathbb{T}_+)] , \\ N_{\mathbb{T}_+}^{-1} &\in [L_2^2(\mathbb{T}_+), L_2^2(\mathbb{T})] . \end{aligned}$$

Let us follow the transformation of coefficients (3.2)

$$M_{\mathbb{T}_+}^{-1} A_{\mathbb{T}} M_{\mathbb{T}_+} = u_1 I_{\mathbb{T}_+} + v_1 \big[S_{\mathbb{T}_+} + V U_{\mathbb{T}_+}\big] \Big(\in \big[L_2^2(\mathbb{T}_+)\big]\Big), \quad V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{3.6}$$

$$(U_{\mathbb{T}_+} f)(t) = \frac{1}{\pi i} \int_{\mathbb{T}_+} \frac{f(\tau)}{\tau + t}\, d\tau, \quad t \in \mathbb{T}_+,$$

$$U_{\mathbb{T}_+} = C_{\mathbb{T}_+} W S_{\mathbb{T}} J_{\mathbb{T}_-} \Big(\in \big[L_2^2(\mathbb{T}_+)\big]\Big).$$

The coefficients of the operator (3.6) are

$$\begin{aligned} u_1 &= C_{\mathbb{T}_+} \begin{bmatrix} a_{\mathbb{T}}(t) & b_{\mathbb{T}}(t) \\ b_{\mathbb{T}}(-t) & a_{\mathbb{T}}(-t) \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} a_{2,11} + a_{2,22} + t a_{2,21} + t^{-1} a_{2,12} & a_{2,11} - a_{2,22} + t a_{2,21} - t^{-1} a_{2,12} \\ a_{2,11} - a_{2,22} - t a_{2,21} + t^{-1} a_{2,12} & a_{2,11} + a_{2,22} - t a_{2,21} - t^{-1} a_{2,12} \end{bmatrix}, \\ v_1 &= C_{\mathbb{T}_+} \begin{bmatrix} c_{\mathbb{T}}(t) & d_{\mathbb{T}}(t) \\ d_{\mathbb{T}}(-t) & c_{\mathbb{T}}(-t) \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} b_{2,11} + b_{2,22} + t b_{2,21} + t^{-1} b_{2,12} & b_{2,11} - b_{2,22} + t b_{2,21} - t^{-1} b_{2,12} \\ b_{2,11} - b_{2,22} - t b_{2,21} + t^{-1} b_{2,12} & b_{2,11} + b_{2,22} - t b_{2,21} - t^{-1} b_{2,12} \end{bmatrix}. \end{aligned}$$

Then, after applying the similarity transformation by Z to the operator (3.6), we get

$$Z^{-1} M_{\mathbb{T}_+}^{-1} A_{\mathbb{T}} M_{\mathbb{T}_+} Z I_{\mathbb{T}_+} = u_2 I_{\mathbb{T}_+} + v_2 \big[S_{\mathbb{T}_+} + \Omega U_{\mathbb{T}_+}\big] \Big(\in \big[L_2^2(\mathbb{T}_+)\big]\Big), \tag{3.7}$$

$$\Omega = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$u_2 = Z^{-1} u_1 Z = \begin{bmatrix} a_{2,11} & t^{-1} a_{2,12} \\ t a_{2,21} & a_{2,22} \end{bmatrix},$$

$$v_2 = Z^{-1} v_1 Z = \begin{bmatrix} b_{2,11} & t^{-1} b_{2,12} \\ t b_{2,21} & b_{2,22} \end{bmatrix}.$$

Then, the operator (3.7) is transformed by the nonsingular matrices $G_{\mathbb{T}_+}^{\pm 1}(t)$, $t \in \mathbb{T}_+$ to the operator

$$\begin{aligned} &\big(G_{\mathbb{T}_+}^{-1} Z^{-1} M_{\mathbb{T}_+}^{-1} A_{\mathbb{T}} M_{\mathbb{T}_+} Z I_{\mathbb{T}_+} G_{\mathbb{T}_+} I_{\mathbb{T}_+} \eta\big)(t) \\ &\qquad = u_3(t)\eta(t) + \frac{v_3(t)}{\pi i} \int_{\mathbb{T}_+} \frac{2\tau}{\tau^2 - t^2}\, \eta(\tau) d\tau, \end{aligned} \tag{3.8}$$

where

$$u_3(t) = \begin{bmatrix} a_{2,11} & a_{2,12} \\ a_{2,21} & a_{2,22} \end{bmatrix}, \quad v_3(t) = \begin{bmatrix} b_{2,11} & b_{2,12} \\ b_{2,21} & b_{2,22} \end{bmatrix}, \quad t \in \mathbb{T}_+.$$

We represent the matrices $u_3(t), v_3(t)$ in the following form

$$u_3(t) = A_0 \chi_{(0,t_0)} + A_1 \chi_{(t_0,t_1)} + A_2 \chi_{(t_1,\pi)},$$
$$v_3(t) = B_0 \chi_{(0,t_0)} + B_1 \chi_{(t_0,t_1)} + B_2 \chi_{(t_1,\pi)},$$

where the constant matrices $A_0, A_1, A_2, B_0, B_1, B_2$ are constructed based on $a_{2,ij}$, $b_{2,ij}$, $i, j = 1, 2$, according to (3.4).

In the last step, in carrying out the similarity transformation, we apply the operator $N_{\mathbb{T}_+}$ from the right-hand side and the operator $N_{\mathbb{T}_+}^{-1}$ from the left-hand side to the operator (3.8) and obtain the matrix characteristic singular integral operator (3.5): $\mathcal{F}^{-1}A_{\mathbb{T}}\mathcal{F} = D_{\mathbb{T}}, D_{\mathbb{T}} = u_{\mathbb{T}}I_{\mathbb{T}} + v_{\mathbb{T}}S_{\mathbb{T}} \in [L_2^2(\mathbb{T})]$ where the coefficients are piecewise constant matrices given on the unit circle with the points of discontinuity at $t = 0, t = t_0^2, t = t_1^2$,

$$u_{\mathbb{T}}(t) = A_0\chi_{(0,t_0^2)} + A_1\chi_{(t_0^2,t_1^2)} + A_2\chi_{(t_1^2,\pi)}\,,$$
$$v_{\mathbb{T}}(t) = B_0\chi_{(0,t_0^2)} + B_1\chi_{(t_0^2,t_1^2)} + B_2\chi_{(t_1^2,\pi)}\,.$$

The constant matrices A_0 and B_0 are given on the contour $(0, t_0^2)$, A_1 and B_1 – on (t_0^2, t_1^2), A_2 and B_2 – on $(t_1^2, 2\pi)$. □

Now we will obtain conditions for the invertibility of the operator $A_{\mathbb{T}}$.

We start with the formulation of a result concerning the invertibility of characteristic singular integral operators with a certain piecewise constant matrix-function [4].

Let us consider the weight space $L_p(\mathbb{R}, \rho)$, $1 < p < \infty$

$$\rho(t) = (1 + t^2)^{\nu/2} \prod_{k=0}^{N-1} |t - t_k|^{\nu_k}\,, \quad t_k \in \mathbb{R}\,.$$

It is known [5] that the singular integral operator $S_{\mathbb{R}}$ is bounded in $L_p(\mathbb{R}, \rho)$ if and only if

$$-\frac{1}{p} < \nu_k < \frac{p-1}{p}\,, \quad k = 0, \dots, N-1\,; \quad -\frac{1}{p} < -\nu - \sum_{j=0}^{N-1} \nu_j + 1 - \frac{2}{p} < \frac{p-1}{p}\,. \tag{3.9}$$

We will assume that (3.9) is in force.

Suppose $N = 2, t_0 = 0, t_1 = 1$. We obtain the weight space

$$L_p(\mathbb{R}, \varrho),\ \varrho = (1+t^2)^{\nu/2}|t|^{\nu_0}|t-1|^{\nu_1}\,, \quad 1 < p < \infty\,, \quad \varrho = (1+t^2)^{\nu/2}|t|^{\nu_0}|t-1|^{\nu_1}\,,$$

where

$$\nu_2 = 1 - \frac{2}{p} - \nu - \nu_0 - \nu_1\,, \quad -\frac{1}{p} < \nu_k < 1 - \frac{1}{p}\,, \quad k = 0, 1, 2\,.$$

Given two non-singular constant matrices $\mathcal{A}$ and $\mathcal{B}$, following [4] we denote the arguments of the eigenvalues of $\mathcal{A}, \mathcal{A}^{-1}\mathcal{B}$ and $\mathcal{B}^{-1}$ by α_{0k}, α_{1k}, and α_{2k} $(k = 1, 2)$, respectively, and introduce the numbers $\nu_{0k}(\mathcal{A}, \mathcal{B}) = \frac{1}{2\pi}\alpha_{0k}, \nu_{1k}(\mathcal{A}, \mathcal{B}) = \frac{1}{2\pi}\alpha_{1k}$, and $\nu_{2k}(\mathcal{A}, \mathcal{B}) = \frac{1}{2\pi}\alpha_{2k}(k = 1, 2)$. In case the matrices $\mathcal{A}$ and $\mathcal{B}$ have common eigenvectors, let us agree upon attaching the same subscript k to the arguments of the eigenvalues associated with the corresponding eigenvectors. If the matrices $\mathcal{A}$ and $\mathcal{B}$ share (up to linear dependence) exactly one common eigenvector, we shall label the corresponding argument of the eigenvalue by the subscript $k = 2$. We introduce the numbers

$$l_k(\mathcal{A},\mathcal{B}) = \sum_{j=0}^{2} \Big(\nu_{jk}(\mathcal{A},\mathcal{B}) + \big[\delta_{jk}(\mathcal{A},\mathcal{B})\big]\Big)\,, \tag{3.10}$$

$$\delta_{jk}(\mathcal{A},\mathcal{B}) = \frac{1}{p} + \nu_j - \nu_{jk}(\mathcal{A},\mathcal{B})\,, \quad k = 1,2\,; \quad j = 0,1,2\,. \tag{3.11}$$

By $[x]$ we mean the integer part of x.

We formulate a theorem from [4].

Theorem 3.2 ([4], Corollary 2, p. 248). *For the operator* $R(G_{\mathbb{R}}) = P_{\mathbb{R}}^{+} + G_{\mathbb{R}} P_{\mathbb{R}}^{-}$, *generated by a matrix-function* $G_{\mathbb{R}} = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]\chi_{(-\infty,0)} + \mathcal{A}\chi_{(0,1)} + \mathcal{B}\chi_{(1,+\infty)}$, *to be invertible in* $L_p^2(\mathbb{R},\varrho)$, *it is necessary and sufficient that the constant matrices* $\mathcal{A},\mathcal{B}$ *are non-singular, that the numbers* $\delta_{jk}(\mathcal{A},\mathcal{B})$ *are non-integer, and that at least one of the following conditions hold:*

(i) $\mathcal{A}$ *and* $\mathcal{B}$ *have no common eigenvectors and* $l_1(\mathcal{A},\mathcal{B}) = -l_2(\mathcal{A},\mathcal{B})$*;*
(ii) $\mathcal{A}$ *and* $\mathcal{B}$ *do not commute, possess a common eigenvector, and* $l_1(\mathcal{A},\mathcal{B}) = -l_2(\mathcal{A},\mathcal{B}) \geq 0$*;*
(iii) $\mathcal{A}$ *and* $\mathcal{B}$ *commute and* $l_1(\mathcal{A},\mathcal{B}) = l_2(\mathcal{A},\mathcal{B}) = 0$.

Results of [4] allow us to find conditions for the invertibility of the operator $A_{\mathbb{T}}$.

Theorem 3.3. *Let* $\det(A_0+B_0) \neq 0, \det(A_1+B_1) \neq 0, \det(A_2+B_2) \neq 0, \det(A_0-B_0) \neq 0$.

In order that the operator $A_{\mathbb{T}} = a_{\mathbb{T}} I_{\mathbb{T}} + c_{\mathbb{T}} S_{\mathbb{T}} + b_{\mathbb{T}} W_{\mathbb{T}} + d_{\mathbb{T}} S_{\mathbb{T}} W_{\mathbb{T}}$, *with orientation-preserving shift on the unit circle* $(W_{\mathbb{T}}\varphi)(t) = \varphi(-t)$ *and coefficients* (3.2), *be invertible on the space* $L_2(\mathbb{T})$, *it is necessary and sufficient that the matrices*

$$\tilde{\mathcal{A}} = (A_0+B_0)(A_0-B_0)^{-1}(A_1+B_1)^{-1}(A_1-B_1)\,,$$
$$\tilde{\mathcal{B}} = (A_0+B_0)(A_0-B_0)^{-1}(A_2+B_2)^{-1}(A_2-B_2)\,.$$

have the following properties

(a) $\det\tilde{\mathcal{A}} \neq 0, \det\tilde{\mathcal{B}} \neq 0$;
(b) *the numbers* $\delta_{jk}(\tilde{\mathcal{A}},\tilde{\mathcal{B}})$ *are not integers* $k = 1,2; j = 0,1,2$;
(c) *for the pair* $\tilde{\mathcal{A}},\tilde{\mathcal{B}}$ *one of the following three conditions* (i), (ii), (iii) *is fulfilled:*
 (i) $\tilde{\mathcal{A}}$ *and* $\tilde{\mathcal{B}}$ *have no common eigenvectors and* $l_1(\tilde{\mathcal{A}},\tilde{\mathcal{B}}) = -l_2(\tilde{\mathcal{A}},\tilde{\mathcal{B}})$*;*
 (ii) $\tilde{\mathcal{A}}$ *and* $\tilde{\mathcal{B}}$ *do not commute, possess a common eigenvector and* $l_1(\tilde{\mathcal{A}},\tilde{\mathcal{B}}) = -l_2(\tilde{\mathcal{A}},\tilde{\mathcal{B}}) \geq 0$*;*
 (iii) $\tilde{\mathcal{A}}$ *and* $\tilde{\mathcal{B}}$ *commute and* $l_1(\tilde{\mathcal{A}},\tilde{\mathcal{B}}) = l_2(\tilde{\mathcal{A}},\tilde{\mathcal{B}}) = 0$.

Proof. Using the projectors $P_{\mathbb{T}}^{+} = \frac{1}{2}(I_{\mathbb{T}} + S_{\mathbb{T}})$ and $P_{\mathbb{T}}^{-} = \frac{1}{2}(I_{\mathbb{T}} - S_{\mathbb{T}})$, we rewrite the operator (3.5):

$$D_{\mathbb{T}} = (u_{\mathbb{T}} + v_{\mathbb{T}})P_{\mathbb{T}}^{+} + (u_{\mathbb{T}} - v_{\mathbb{T}})P_{\mathbb{T}}^{-}\,.$$

We assume that $\det(u_{\mathbb{T}} + v_{\mathbb{T}}) \neq 0$, or, using formulas (3.3),

$$\det(A_0 + B_0) \neq 0\,, \quad \det(A_1 + B_1) \neq 0\,, \quad \det(A_2 + B_2) \neq 0\,.$$

Applying the matrix $(u_{\mathbb{T}} + v_{\mathbb{T}})^{-1}$ from the left-hand side to the operator $D_{\mathbb{T}}$, we get

$$R(\mathcal{G}_{\mathbb{T}}) = P^+_{\mathbb{T}} + \mathcal{G}_{\mathbb{T}} P^-_{\mathbb{T}}\,, \quad \mathcal{G}_{\mathbb{T}}(t) = (u_{\mathbb{T}} + v_{\mathbb{T}})^{-1}(u_{\mathbb{T}} - v_{\mathbb{T}})\,, \quad R(\mathcal{G}_{\mathbb{T}}) \in \left[L^2_2(\mathbb{T}), L^2_2(\mathbb{T})\right].$$

Using the operators $\Lambda^{-1} \in [L^2_2(\mathbb{T}), L^2_2(\mathbb{R})]$, $\Lambda \in [L^2_2(\mathbb{R}), L^2_2(\mathbb{T})]$,

$$\left(\Lambda^{-1}\varphi\right)(x) = \frac{2i}{i+x}\varphi\left(-\frac{i-x}{i+x}\right), \quad (\Lambda f)(t) = \frac{1}{1-t} f\left(i\frac{1+t}{1-t}\right),$$

we reduce the operator from the unit circle $\mathbb{T}$ to the real axis $\Lambda^{-1}R(\mathcal{G}_{\mathbb{T}})\Lambda = R(\mathcal{G}_{\mathbb{R}})$,

$$R(\mathcal{G}_{\mathbb{R}}) = \Lambda^{-1}(P^+_{\mathbb{T}} + \mathcal{G}_{\mathbb{T}} P^-_{\mathbb{T}})\Lambda = P^+_{\mathbb{R}} + \mathcal{G}_{\mathbb{R}} P^-_{\mathbb{R}}\,, \quad R(\mathcal{G}_{\mathbb{R}}) \in \left[L^2_2(\mathbb{R}), L^2_2(\mathbb{R})\right].$$

Here the matrix $\mathcal{G}_{\mathbb{R}}(x) = (u_{\mathbb{R}}(x) + v_{\mathbb{R}}(x))^{-1}(u_{\mathbb{R}}(x) - v_{\mathbb{R}}(x))$ and

$$u_{\mathbb{R}}(x) = A_0\chi_{(-\infty,x_0)} + A_1\chi_{(x_0,x_1)} + A_2\chi_{(x_1,\infty)}\,,$$
$$v_{\mathbb{R}}(x) = B_0\chi_{(-\infty,x_0)} + B_1\chi_{(x_0,x_1)} + B_2\chi_{(x_1,\infty)}\,,$$

$x_0 = i\frac{1+t_0^2}{1-t_0^2}, x_1 = i\frac{1+t_1^2}{1-t_1^2}$. The matrix $\mathcal{G}_{\mathbb{R}}(x), x \in \mathbb{R}$, is expressed in terms of the constant matrix-functions of the input coefficients $a_{2,ij}(t), b_{2,ij}(t),\ t \in \mathbb{T}$, in the following form:

$$\mathcal{G}_{\mathbb{R}}(x) = C_0\chi_{(-\infty,x_0)} + C_1\chi_{(x_0,x_1)} + C_2\chi_{(x_1+\infty)}\,,$$
$$C_j = (A_j + B_j)^{-1}(A_j - B_j)\,, \quad j = 0, 1, 2\,.$$

The matrix $\mathcal{G}_{\mathbb{R}}(x), x \in \mathbb{R}$ is a piecewise constant matrix function with three values and points of discontinuity at $x = x_0, x = x_1$.

In the case that the matrix C_0 is non singular – that is, the matrix $A_0 - B_0$ is non singular – we have

$$\tilde{\mathcal{A}} = C_0^{-1}C_1 = (A_0 + B_0)(A_0 - B_0)^{-1}(A_1 + B_1)^{-1}(A_1 - B_1)\,,$$
$$\tilde{\mathcal{B}} = C_0^{-1}C_2 = (A_0 + B_0)(A_0 - B_0)^{-1}(A_2 + B_2)^{-1}(A_2 - B_2)\,.$$

Using the matrices $\tilde{\mathcal{A}}, \tilde{\mathcal{B}}$, we construct the numbers $\delta_{jk}(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$, $l_k(\tilde{\mathcal{A}}, \tilde{\mathcal{B}})$ according to (3.10), (3.11).

Applying Theorem 2.2 to the operator $R(\mathcal{G}_{\mathbb{R}})$, we obtain the proof. $\square$

4. Matrix characteristic singular integral operators with piecewise constant coefficients of a special structure which have four values

An operator $Q \in [L_p(\mathbb{R}, \rho_Q)]$ is called involutive if $Q^2 = I, Q \neq \pm I$. Results of investigations of the involutive operators can be found in [6]. We introduce the operator

$$\left(Q_{\mathbb{R}}\,\varphi\right)(x) = \frac{\sqrt{\delta^2 + \beta}}{x - \delta}\varphi\left[\alpha(x)\right], \quad \alpha(x) = \frac{\delta x + \beta}{x - \delta}\,, \quad \delta^2 + \beta > 0\,.$$

In the space $L_p(\mathbb{R}, \rho_Q)$ consider operator:

$$B_{\mathbb{R}} = aI_{\mathbb{R}} + bQ_{\mathbb{R}} + cS_{\mathbb{R}} + dQ_{\mathbb{R}}S_{\mathbb{R}}\,, \tag{4.1}$$

with piecewise constant coefficients

$$\begin{aligned} a(x) &= a_{-2}\chi_{(-\infty,x_1)}(x) + a_{-1}\chi_{(x_1,\delta)}(x) + a_{+1}\chi_{(\delta,x_2)}(x) + a_{+2}\chi_{(x_2,+\infty)}(x)\,, \\ b(x) &= b_{-2}\chi_{(-\infty,x_1)}(x) + b_{-1}\chi_{(x_1,\delta)}(x) + b_{+1}\chi_{(\delta,x_2)}(x) + b_{+2}\chi_{(x_2,+\infty)}(x)\,, \\ c(x) &= c_{-2}\chi_{(-\infty,x_1)}(x) + c_{-1}\chi_{(x_1,\delta)}(x) + c_{+1}\chi_{(\delta,x_2)}(x) + c_{+2}\chi_{(x_2,+\infty)}(x)\,, \\ d(x) &= d_{-2}\chi_{(-\infty,x_1)}(x) + d_{-1}\chi_{(x_1,\delta)}(x) + d_{+1}\chi_{(\delta,x_2)}(x) + d_{+2}\chi_{(x_2,+\infty)}(x)\,, \end{aligned} \tag{4.2}$$

where

$$x_1 = \delta - \sqrt{\delta^2 + \beta}\,, \quad x_2 = \delta + \sqrt{\delta^2 + \beta}, a_j, b_j, c_j, d_j\,, \quad j = -2, -1, +1, +2\,,$$

are real numbers. The function $\alpha(x)$ is a Carleman shift: $\alpha[\alpha(x)] = x, x \in \mathbb{R}$ and $\alpha(x_1) = x_2, \alpha(\delta) = \infty$.

The restrictions on the weight function $\rho_Q(x) = \prod_{j=1}^{4} |x - x_j|^{\mu_j}$, $x_3 = \delta$, $x_4 = i$ are defined by formulas (3.9): $\frac{-1}{p} < \mu_j < \frac{p-1}{p}, j = 1, 2, 3;$ $\frac{-1}{p} < \sum_{j=1}^{4} \mu_j < \frac{p-1}{p}$; and

$$\mu_1 = \mu_2\,, \quad \mu_3 = -\sum_{j=1}^{4} \mu_j + \frac{p-2}{p}\,. \tag{4.3}$$

The conditions (3.9) ensure the boundedness of $S_{\mathbb{R}}$ while the conditions (4.3) ensure the boundedness of $Q_{\mathbb{R}}$ in the space $L_p(\mathbb{R}, \rho_Q)$.

The operator $Q_{\mathbb{R}}$ is an involutive operator and possesses an additional property: $Q_{\mathbb{R}}S_{\mathbb{R}} = -S_{\mathbb{R}}Q_{\mathbb{R}}$.

Let us introduce the operators

$$(\Theta\varphi)(x) = \frac{x_2 - x_1}{x_2 - x}\varphi\left(\frac{x - x_1}{x_2 - x}\right), \quad (\Theta^{-1}\varphi)(x) = \frac{1}{x+1}\varphi\left(\frac{x_2 x + x_1}{x+1}\right).$$

Theorem 4.1. *The singular integral operator with an involutive operator, generated by a Carleman linear-fractional orientation-reversing shift,*

$$B_{\mathbb{R}} = aI_{\mathbb{R}} + bQ_{\mathbb{R}} + cS_{\mathbb{R}} + dQ_{\mathbb{R}}S_{\mathbb{R}}\,,$$

acting on the space $L_2(\mathbb{R}, \rho_Q)$ is reduced to the matrix characteristic singular integral operator

$$\begin{aligned} D^1_{\mathbb{R}} &= u_{\mathbb{R}} I_{\mathbb{R}} + v_{\mathbb{R}} S_{\mathbb{R}}\,, \quad u_{\mathbb{R}} = \chi_{\mathbb{R}_-} + J_{\mathbb{R}_-} u_{\mathbb{R}_+}\,, \\ v_{\mathbb{R}} &= J_{\mathbb{R}_-} v_{\mathbb{R}_+}\,, \quad D^1_{\mathbb{R}} \in \left[L^2_2(\mathbb{R}, \varrho)\right], \end{aligned} \tag{4.4}$$

where the weight is $\varrho(x) = |x|^{\frac{1}{2}(\mu_1 - \frac{1}{p})}|x-1|^{\mu_3}|x-i|^{\frac{1}{2}\mu_4}$ *and the coefficients are piecewise constant matrices given on the real axis and with three values:*

$$\begin{aligned} u_{\mathbb{R}_+} &= Z^{-1}u_1 Z \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + Z^{-1}v_1 Z \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \\ v_{\mathbb{R}_+} &= Z^{-1}u_1 Z \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + Z^{-1}v_1 Z \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} u_1(x) &= \chi_{(0,1)}(x) \begin{bmatrix} a_{-1} & -b_{-1} \\ -b_{-2} & a_{-2} \end{bmatrix} + \chi_{(1,+\infty)}(x) \begin{bmatrix} a_{+1} & -b_{+1} \\ -b_{+2} & a_{+2} \end{bmatrix}, \\ v_1(x) &= \chi_{(0,1)}(x) \begin{bmatrix} c_{-1} & d_{-1} \\ -d_{-2} & -c_{-2} \end{bmatrix} + \chi_{(1,+\infty)}(x) \begin{bmatrix} c_{+1} & d_{+1} \\ -d_{+2} & -c_{+2} \end{bmatrix}, \quad x \in \mathbb{R}_+ . \end{aligned} \tag{4.6}$$

Proof. Applying the operator equality (2.5) to $B_{\mathbb{R}}$, we reduce it to the characteristic operator:

$$N_{\mathbb{R}_+}^{-1}\, Z^{-1}\, M_{\mathbb{R}}^{-1}\, \Theta^{-1}\, B_{\mathbb{R}}\, \Theta\, M_{\mathbb{R}}\, Z\, P\, N_{\mathbb{R}_+} \;=\; D_{\mathbb{R}_+} .$$

Let us follow the transformation of coefficients.

Carrying out the transformation $B_{\mathbb{R}} \longmapsto \Theta^{-1}B_{\mathbb{R}}\Theta$, we obtain the operator $A_{\mathbb{R}} \in [L_p(\mathbb{R}, \rho_W)]$, $\rho_W(x) = |x|^{\mu_1}|x^2-1|^{\mu_3}|x-i|^{\mu_4}$,

$$A_{\mathbb{R}} = a_2 I_{\mathbb{R}} + b_2 S_{\mathbb{R}} + a_1 W_{\mathbb{R}} + b_1 S_{\mathbb{R}} W_{\mathbb{R}} = a_2 I_{\mathbb{R}} + a_1 W_{\mathbb{R}} + (b_2 I_{\mathbb{R}} - b_1 W_{\mathbb{R}}) S_{\mathbb{R}} ,$$

with the reflection $(W_{\mathbb{R}}\varphi)(x) = \varphi(-x)$ and the coefficients

$$\begin{aligned} a_1(x) &= -b\big[\gamma(x)\big] = -\big[b_{+2}\chi_{(-\infty,-1)} + b_{-2}\chi_{(-1,0)} + b_{-1}\chi_{(0,1)} + b_{+1}\chi_{(1,+\infty)}\big] , \\ a_2(x) &= a\big[\gamma(x)\big] = \big[a_{+2}\chi_{(-\infty,-1)} + a_{-2}\chi_{(-1,0)} + a_{-1}\chi_{(0,1)} + a_{+1}\chi_{(1,+\infty)}\big] , \\ b_1(x) &= d\big[\gamma(x)\big] = \big[d_{+2}\chi_{(-\infty,-1)} + d_{-2}\chi_{(-1,0)} + d_{-1}\chi_{(0,1)} + d_{+1}\chi_{(1,+\infty)}\big] , \\ b_2(x) &= c\big[\gamma(x)\big] = \big[c_{+2}\chi_{(-\infty,-1)} + c_{-2}\chi_{(-1,0)} + c_{-1}\chi_{(0,1)} + c_{+1}\chi_{(1,+\infty)}\big] , \end{aligned}$$

where

$$\gamma(x) = \frac{x_2 x + x_1}{x+1} .$$

After applying the similarity transformation to the operator $A_{\mathbb{R}}$, we get the operator

$$M_{\mathbb{R}}^{-1} A_{\mathbb{R}} M_{\mathbb{R}} = u_1 I_{\mathbb{R}_+} + v_1 \sum_{k=1}^{2} (-1)^k V^{-k} U_{k,\mathbb{R}_+}$$

with the following coefficients

$$\begin{aligned}
u_1(x) &= \sum_{k=1}^{2} diag\big[C_{\mathbb{R}_+} a_k, C_{\mathbb{R}_+}(W_{\mathbb{R}} a_k)\big] V^k \\
&= \chi_{(0,1)}(x) \begin{bmatrix} a_{-1} & -b_{-1} \\ -b_{-2} & a_{-2} \end{bmatrix} + \chi_{(1,+\infty)}(x) \begin{bmatrix} a_{+1} & -b_{+1} \\ -b_{+2} & a_{+2} \end{bmatrix}, \\
v_1(x) &= C_{\mathbb{R}_+} diag\big[-b_1 V + b_2, W_{\mathbb{R}}(-b_1 V + b_2)\big]\Omega \\
&= \chi_{(0,1)}(x) \begin{bmatrix} c_{-1} & d_{-1} \\ -d_{-2} & -c_{-2} \end{bmatrix} + \chi_{(1,+\infty)}(x) \begin{bmatrix} c_{+1} & d_{+1} \\ -d_{+2} & -c_{+2} \end{bmatrix}, \quad x \in \mathbb{R}_+ .
\end{aligned}$$

The coefficients $u_1(x)$ and $v_1(x)$ are piecewise constant matrix functions with two values: on $(0,1)$ and on $(1,+\infty)$.

The operators Z, Z^{-1} and P do not affect the point of discontinuity at $x = 1$. The operator N^{-1} does not affect the coefficients. Thus, having applied the operator equality (2.5), we arrive at the operator

$$D_{\mathbb{R}_+} = u_{\mathbb{R}_+} I_{\mathbb{R}_+} + v_{\mathbb{R}_+} S_{\mathbb{R}_+} \in \big[L_p^2(\mathbb{R}_+, \varrho)\big]$$

with the coefficients (4.5).

Extending the operator $D_{\mathbb{R}_+}$ to the entire real axis

$$D^1_{\mathbb{R}} = u_{\mathbb{R}} I_{\mathbb{R}} + v_{\mathbb{R}} S_{\mathbb{R}} , \quad u_{\mathbb{R}} = \chi_{\mathbb{R}_-} + J_{\mathbb{R}_-} u_{\mathbb{R}_+} , \quad v_{\mathbb{R}} = J_{\mathbb{R}_-} v_{\mathbb{R}_+} \in \big[L_p^2(\mathbb{R}, \varrho)\big] ,$$

(note that the property of being invertible is retained), we complete the proof of the theorem. □

Let us consider the weight space $L_p(\mathbb{R}, \rho_W), p \geq 1$,

$$(\rho_W)(x) = \prod_{j=1}^{4} |x - x_j|^{\mu_j} , \quad x_1 = -1 , \quad x_2 = 1 , \quad x_3 = 0 , \quad x_4 = i$$

with the norm $\|f\|_{L_p(\mathbb{R},\rho_W)} = \|\rho_W f\|_{L_p(\mathbb{R})}$.

Assuming that the following conditions hold:

$$\frac{-1}{p} < \mu_j < \frac{p-1}{p} , \quad j = 1, 2, 3 ; \quad \frac{-1}{p} < \sum_{j=1}^{4} \mu_j < \frac{p-1}{p} ; \quad \mu_1 = \mu_2 , \tag{4.7}$$

we get from [5] that the operators

$$S_{\mathbb{R}} \in \big[L_p(\mathbb{R}, \rho_W)\big] ; \quad W_{\mathbb{R}} \in \big[L_p(\mathbb{R}, \rho_W)\big] ,$$

where

$$(W_{\mathbb{R}} \varphi)(x) = \varphi(-x) .$$

In the space $L_p^2(\mathbb{R}, \rho_W)$ consider the operator $\mathcal{D}_{\mathbb{R}} = u I_{\mathbb{R}} + v S_{\mathbb{R}}$ with coefficients which are piecewise constant matrix functions with three points of discontinuity at $x=-1, x=0, x=1$:

$$
\begin{aligned}
u = &\begin{bmatrix} a_{-2} & b_{-2} \\ b_{+2} & a_{+2} \end{bmatrix} \chi_{(-\infty,-1)} + \begin{bmatrix} a_{-1} & b_{-1} \\ b_{+1} & a_{+1} \end{bmatrix} \chi_{-1,0)} \\
&+ V \begin{bmatrix} a_{-1} & b_{-1} \\ b_{+1} & a_{+1} \end{bmatrix} V \chi_{(0,+1)} + V \begin{bmatrix} a_{-2} & b_{-2} \\ b_{+2} & a_{+2} \end{bmatrix} V \chi_{(1,\infty)}\,, \\
v = &\begin{bmatrix} c_{-2} & -d_{-2} \\ d_{+2} & -c_{+2} \end{bmatrix} \chi_{(-\infty,-1)} + \begin{bmatrix} c_{-1} & -d_{-1} \\ d_{+1} & -c_{+1} \end{bmatrix} \chi_{(-1,0)} \\
&- V \begin{bmatrix} c_{-1} & -d_{-1} \\ d_{+1} & -c_{+2} \end{bmatrix} V \chi_{(0,1)} - V \begin{bmatrix} c_{-2} & -d_{-2} \\ d_{+2} & -c_{+2} \end{bmatrix} V \chi_{(1,\infty)}\,,
\end{aligned} \tag{4.8}
$$

where

$$
V = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
$$

The coefficients (4.8) are selected so that, after applying the Gohberg–Krupnik matrix equality $\mathcal{D}_{\mathbb{R}} = H\left[\begin{smallmatrix} B_+ & 0 \\ 0 & B_- \end{smallmatrix}\right]H^{-1}$ we would have scalar operators B_+, B_- considered in Theorem 4.1.

In this section, conditions for the invertibility of the operator $\mathcal{D}_{\mathbb{R}}$ in the space $L_p^2(\mathbb{R}, \rho_W)$ are obtained.

We construct the matrices

$$
\begin{aligned}
\mathcal{A}^{\pm} = &-\det{}^{-1} \begin{bmatrix} a_{-1}+c_{-1} & \mp b_{-1} \pm d_{-1} \\ \mp b_{-2} \mp d_{-2} & a_{-2}-c_{-2} \end{bmatrix} \\
&\cdot Z^{-1} \begin{bmatrix} a_{-2}-c_{-2} & \pm b_{-1} \mp d_{-1} \\ \pm b_{-2} \pm d_{-2} & a_{-1}+c_{-1} \end{bmatrix} \begin{bmatrix} a_{-1}-c_{-1} & \mp b_{-1} \mp d_{-1} \\ \mp b_{-2} \pm d_{-2} & a_{-2}+c_{-2} \end{bmatrix} Z\Omega\,, \\
\mathcal{B}^{\pm} = &-\det{}^{-1} \begin{bmatrix} a_{+1}+c_{+1} & \mp b_{+1} \pm d_{+1} \\ \mp b_{+2} \mp d_{+2} & a_{+2}-c_{+2} \end{bmatrix} \\
&\cdot Z^{-1} \begin{bmatrix} a_{+2}-c_{+2} & \pm b_{+1} \mp d_{+1} \\ \pm b_{+2} \pm d_{+2} & a_{+1}+c_{+1} \end{bmatrix} \begin{bmatrix} a_{+1}-c_{+1} & \mp b_{+1} \mp d_{+1} \\ \mp b_{+2} \pm d_{+2} & a_{+2}+c_{+2} \end{bmatrix} Z\Omega\,,
\end{aligned} \tag{4.9}
$$

where

$$
Z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
$$

Using definitions (3.10), (3.11) of Section 2, we introduce the constants

$$
l_k^{\pm} = l_k(\mathcal{A}^{\pm}, \mathcal{B}^{\pm})\,, \quad \delta_{jk}^{\pm} = \delta_{jk}(\mathcal{A}^{\pm}, \mathcal{B}^{\pm})\,.
$$

Theorem 4.2. *Let*

$$
\det \begin{bmatrix} a_{-1}+c_{-1} & \mp b_{-1} \pm d_{-1} \\ \mp b_{-2} \mp d_{-2} & a_{-2}-c_{-2} \end{bmatrix} \neq 0\,,
$$

$$
\det \begin{bmatrix} a_{+1}+c_{+1} & \mp b_{+1} \pm d_{+1} \\ \mp b_{+2} \mp d_{+2} & a_{+2}-c_{+2} \end{bmatrix} \neq 0\,.
$$

In order that the operator $\mathcal{D}_{\mathbb{R}}$,

$$
\mathcal{D}_{\mathbb{R}} = \begin{bmatrix} a(x) & b(x) \\ b(-x) & a(-x) \end{bmatrix} I_{\mathbb{R}} + \begin{bmatrix} c(x) & -d(x) \\ d(-x) & -c(-x) \end{bmatrix} S_{\mathbb{R}}\,,
$$

with piecewise constant coefficients and points of discontinuity at $x=-1, x=0$, $x=1$,

$$\begin{aligned}
a(x) &= a_{-2}\chi_{(-\infty,-1)}(x) + a_{-1}\chi_{(-1,0)}(x) + a_{+1}\chi_{(0,1)}(x) + a_{+2}\chi_{(1,+\infty)}(x)\,,\\
b(x) &= b_{-2}\chi_{(-\infty,-1)}(x) + b_{-1}\chi_{(-1,0)}(x) + b_{+1}\chi_{(0,1)}(x) + b_{+2}\chi_{(1,+\infty)}(x)\,,\\
c(x) &= c_{-2}\chi_{(-\infty,-1)}(x) + c_{-1}\chi_{(-1,0)}(x) + c_{+1}\chi_{(0,1)}(x) + c_{+2}\chi_{(1,+\infty)}(x)\,,\\
d(x) &= d_{-2}\chi_{(-\infty,-1)}(x) + d_{-1}\chi_{(-1,0)}(x) + d_{+1}\chi_{(0,1)}(x) + d_{+2}\chi_{(1,+\infty)}(x)\,,
\end{aligned}$$

be invertible on the space $L_p(\mathbb{R},\rho_W)$, *it is necessary and sufficient that the matrices* $\mathcal{A}^+,\mathcal{B}^+$ *and* $\mathcal{A}^-,\mathcal{B}^-$ *have the following properties:*

(a) $\det\mathcal{A}^+\neq 0, \det\mathcal{B}^+\neq 0$ *and* $\det\mathcal{A}^-\neq 0, \det\mathcal{B}^-\neq 0$

(b) *the numbers* $\delta^+_{jk}=\delta_{jk}(\mathcal{A}^+,\mathcal{B}^+)$ *and* $\delta^-_{jk}=\delta_{jk}(\mathcal{A}^-,\mathcal{B}^-)$ *are not integers* $k=1,2; j=0,1,2$.

(c) *for the pair* $\mathcal{A}^+,\mathcal{B}^+$ *and for the pair* $\mathcal{A}^-,\mathcal{B}^-$ *one of the following three conditions* (i), (ii), (iii) *is fulfilled:*

  (i) $\mathcal{A}^\pm$ *and* $\mathcal{B}^\pm$ *have no common eigenvectors and* $l_1(\mathcal{A}^\pm,\mathcal{B}^\pm)=-l_2(\mathcal{A}^\pm,\mathcal{B}^\pm)$;

  (ii) $\mathcal{A}^\pm$ *and* $\mathcal{B}^\pm$ *do not commute, posses a common eigenvector and* $l_1(\mathcal{A}^\pm,\mathcal{B}^\pm)$ $=-l_2(\mathcal{A}^\pm,\mathcal{B}^\pm)\geq 0$;

  (iii) $\mathcal{A}^\pm$ *and* $\mathcal{B}^\pm$ *commute and* $l_1(\mathcal{A}^\pm,\mathcal{B}^\pm)=l_2(\mathcal{A}^\pm,\mathcal{B}^\pm)=0$.

Proof. By the Gohberg–Krupnik matrix equality (1.1)

$$H\begin{bmatrix} aI_{\mathbb{R}}+bW_{\mathbb{R}}I_{\mathbb{R}}+cS_{\mathbb{R}}+dW_{\mathbb{R}}S_{\mathbb{R}} & 0\\ 0 & aI_{\mathbb{R}}-bW_{\mathbb{R}}I_{\mathbb{R}}+cS_{\mathbb{R}}-dW_{\mathbb{R}}S_{\mathbb{R}}\end{bmatrix}H^{-1}=\mathcal{D}_{\mathbb{R}}\,,$$

where

$$H=\frac{1}{2}\begin{bmatrix} I_{\mathbb{R}} & I_{\mathbb{R}}\\ W_{\mathbb{R}} & -W_{\mathbb{R}}\end{bmatrix},\quad H^{-1}=\begin{bmatrix} I_{\mathbb{R}} & W_{\mathbb{R}}\\ I_{\mathbb{R}} & -W_{\mathbb{R}}\end{bmatrix},$$

the singular integral operator $\mathcal{D}_{\mathbb{R}}$ is invertible on the space $L_p(\mathbb{R},\rho_W)$, if and only if the operators

$$B=B_+=aI_{\mathbb{R}}+bW_{\mathbb{R}}+cS_{\mathbb{R}}+dW_{\mathbb{R}}S_{\mathbb{R}}\,,\quad B_-=aI_{\mathbb{R}}-bW_{\mathbb{R}}+cS_{\mathbb{R}}-dW_{\mathbb{R}}S_{\mathbb{R}}$$

are invertible operators on the space $L_p(\mathbb{R},\rho_W)$.

Applying the operator equality (2.5) to B^+ and B^-, we have

$$\mathcal{D}^\pm_{\mathbb{R}_+}=\mathcal{H}B^\pm\mathcal{E}=u^\pm_{\mathbb{R}_+}I_{\mathbb{R}_+}+v^\pm_{\mathbb{R}_+}S_{\mathbb{R}_+}\,,\quad \mathcal{D}^\pm_{\mathbb{R}_+}\in\left[L^2_p(\mathbb{R}_+,\varrho)\right].$$

The weight ρ_W is transformed to

$$\begin{aligned}
\varrho(x) &= |x|^{\nu_0}|x-1|^{\nu_1}|x-i|^{\nu}\,,\\
\nu_0 &= \frac{1}{2}\left(\mu_3-\frac{1}{p}\right),\quad \nu_1=\mu_1=\mu_2\,,\quad \nu=\frac{1}{2}\mu_4\,.
\end{aligned}$$

From formulas (2.6), (2.7) the coefficients of the operator $\mathcal{D}^{\pm}_{\mathbb{R}_+}$ are

$$u^{\pm}_{\mathbb{R}_+}(t) = \frac{1}{2}\begin{bmatrix} (c_{-1} \pm d_{-1}) - (c_{-2} \pm d_{-2}) & (a_{-1} \pm b_{-1}) - (a_{-2} \pm b_{-2}) \\ (c_{-1} \pm d_{-1}) + (c_{-2} \pm d_{-2}) & (a_{-1} \pm b_{-1}) + (a_{-2} \pm b_{-2}) \end{bmatrix} \chi_{(0,1)}(t)$$
$$+ \frac{1}{2}\begin{bmatrix} (c_{+1} \pm d_{+1}) - (c_{+2} \pm d_{+2}) & (a_{+1} \pm b_{+1}) - (a_{+2} \pm b_{+2}) \\ (c_{+1} \pm d_{+1}) + (c_{+2} \pm d_{+2}) & (a_{+1} \pm b_{+1}) + (a_{+2} \pm b_{+2}) \end{bmatrix} \chi_{(1,\infty)}(t)\,,$$
$$v^{\pm}_{\mathbb{R}_+}(t) = \frac{1}{2}\begin{bmatrix} (a_{-1} \mp b_{-1}) + (a_{-2} \mp b_{-2}) & (c_{-1} \mp d_{-1}) + (c_{-2} \mp d_{-2}) \\ (a_{-1} \mp b_{-1}) - (a_{-2} \mp b_{-2}) & (c_{-1} \mp d_{-1}) - (c_{-2} \mp d_{-2}) \end{bmatrix} \chi_{(0,1)}(t)$$
$$+ \frac{1}{2}\begin{bmatrix} (a_{+1} \mp b_{+1}) + (a_{+2} \mp b_{+2}) & (c_{+1} \mp d_{+1}) + (c_{+2} \mp d_{+2}) \\ (a_{+1} \mp b_{+1}) - (a_{+2} \mp b_{+2}) & (c_{+1} \mp d_{+1}) - (c_{+2} \mp d_{+2}) \end{bmatrix} \chi_{(1,\infty)}(t)\,.$$

Extend the operator $\mathcal{D}^{\pm}_{\mathbb{R}_+}$ to the entire real axis

$$\mathcal{D}^{\pm}_{\mathbb{R}} = J_{\mathbb{R}_+} C_{\mathbb{R}_-} + J_{\mathbb{R}_-} \mathcal{D}^{\pm}_{\mathbb{R}_+} C_{\mathbb{R}_+}\,, \quad \mathcal{D}^{\pm}_{\mathbb{R}} \in \left[L^2_p(\mathbb{R}, \varrho)\right],$$

and rewrite $\mathcal{D}^{\pm}_{\mathbb{R}}$ using the projections

$$\mathcal{D}^{\pm}_{\mathbb{R}} = \mathcal{U}^{\pm}_{\mathbb{R}} P^{+}_{\mathbb{R}} + \mathcal{V}^{\pm}_{\mathbb{R}} P^{-}_{\mathbb{R}}\,,$$

where

$$\mathcal{U}^{\pm}_{\mathbb{R}} = \chi_{\mathbb{R}_-} + J_{\mathbb{R}_-}(u^{\pm}_{\mathbb{R}} + v^{\pm}_{\mathbb{R}_+})\,, \quad \mathcal{V}^{\pm}_{\mathbb{R}} = \chi_{\mathbb{R}_-} + J_{\mathbb{R}_-}(u^{\pm}_{\mathbb{R}} - v^{\pm}_{\mathbb{R}_+})\,.$$

The matrices $\mathcal{U}^{\pm}_{\mathbb{R}} = u^{\pm}_{\mathbb{R}_+}(t) + v^{\pm}_{\mathbb{R}_+}(t)$ and $\mathcal{V}^{\pm}_{\mathbb{R}} = u^{\pm}_{\mathbb{R}_+}(t) - v^{\pm}_{\mathbb{R}_+}(t)$ have the following form

$$\mathcal{U}^{\pm}_{\mathbb{R}} = \chi_{\mathbb{R}_-} + J_{\mathbb{R}_-}\Pi\left\{ \begin{bmatrix} a_{-1} + c_{-1} & \mp b_{-1} \pm d_{-1} \\ \mp b_{-2} \mp d_{-2} & a_{-2} - c_{-2} \end{bmatrix} \chi_{(0,1)} + \begin{bmatrix} a_{+1} + c_{+1} & \mp b_{+1} \pm d_{+1} \\ \mp b_{+2} \mp d_{+2} & a_{+2} - c_{+2} \end{bmatrix} \chi_{(1,\infty)} \right\}\Pi\,,$$

$$\mathcal{V}^{\pm}_{\mathbb{R}} = \chi_{\mathbb{R}_-} - J_{\mathbb{R}_-}\Pi\left\{ \begin{bmatrix} a_{-1} - c_{-1} & \mp b_{-1} \mp d_{-1} \\ \mp b_{-2} \pm d_{-2} & a_{-2} + c_{-2} \end{bmatrix} \chi_{(0,1)} + \begin{bmatrix} a_{+1} - c_{+1} & \mp b_{+1} \mp d_{+1} \\ \mp b_{+2} \pm d_{+2} & a_{+2} + c_{+2} \end{bmatrix} \chi_{(1,\infty)} \right\}\Pi\Omega\,.$$

We assume that $det[u^{\pm}_{\mathbb{R}_+}(t) + v^{\pm}_{\mathbb{R}_+}(t)] \neq 0$, or, rewriting in an equivalent form,

$$det\left\{ \begin{bmatrix} a_{-1} + c_{-1} & \mp b_{-1} \pm d_{-1} \\ \mp b_{-2} \mp d_{-2} & a_{-2} - c_{-2} \end{bmatrix} \chi_{(0,1)} + \begin{bmatrix} a_{+1} + c_{+1} & \mp b_{+1} \pm d_{+1} \\ \mp b_{+2} \mp d_{+2} & a_{+2} - c_{+2} \end{bmatrix} \chi_{(1,\infty)} \right\} \neq 0\,.$$

Having calculated the matrix $\mathcal{G}^{\pm} = (\mathcal{U}^{\pm}_{\mathbb{R}})^{-1}\mathcal{V}^{\pm}_{\mathbb{R}}$, we obtain

$$\mathcal{G}^{\pm} = \chi_{\mathbb{R}_-} + \mathcal{A}^{\pm}\chi_{(0,1)} + \mathcal{B}^{\pm}\chi_{(1,\infty)}\,,$$

where the matrices $\mathcal{A}^{\pm}$ and $\mathcal{B}^{\pm}$ are given by formulas (4.9). The matrix $\mathcal{G}^{\pm}(x)$, $x \in \mathbb{R}$, is a piecewise constant matrix function with three values and points of discontinuity at $x = 0, x = 1$. The operator $\mathcal{R}(\mathcal{G}^{\pm}) = P^{+}_{\mathbb{R}} + \mathcal{G}^{\pm}P^{-}_{\mathbb{R}}$ is invertible on the space $L^2_p(\mathbb{R}, \varrho)$ if and only if the operator $\mathcal{D}^{\pm}_{\mathbb{R}_+}$ is invertible on the space $L^2_p(\mathbb{R}_+, \varrho)$.

Applying Theorem 2.2 to the operator $\mathcal{R}(\mathcal{G}^{\pm})$ we complete the proof of the theorem. □

Corollary 4.3. *In order that the operator $B_{\mathbb{R}}$ with piecewise constant coefficients and the points of discontinuity at $x = -1, x = 0,\ x = 1$ be invertible on the space $L_p(\mathbb{R}, \rho_Q)$ it is necessary and sufficient that the matrices*

$$\begin{aligned}\mathcal{A} &= -\det{}^{-1}\begin{bmatrix} a_{-1}+c_{-1} & -b_{-1}+d_{-1} \\ -b_{-2}-d_{-2} & a_{-2}-c_{-2} \end{bmatrix} \\ &\quad \cdot Z^{-1}\begin{bmatrix} a_{-2}-c_{-2} & b_{-1}-d_{-1} \\ b_{-2}+d_{-2} & a_{-1}+c_{-1} \end{bmatrix}\begin{bmatrix} a_{-1}-c_{-1} & -b_{-1}-d_{-1} \\ -b_{-2}+d_{-2} & a_{-2}+c_{-2} \end{bmatrix} Z\Omega\,, \\ \mathcal{B} &= -\det{}^{-1}\begin{bmatrix} a_{+1}+c_{+1} & -b_{+1}+d_{+1} \\ -b_{+2}-d_{+2} & a_{+2}-c_{+2} \end{bmatrix} \\ &\quad \cdot Z^{-1}\begin{bmatrix} a_{+2}-c_{+2} & b_{+1}-d_{+1} \\ b_{+2}+d_{+2} & a_{+1}+c_{+1} \end{bmatrix}\begin{bmatrix} a_{+1}-c_{+1} & -b_{+1}-d_{+1} \\ -b_{+2}+d_{+2} & a_{+2}+c_{+2} \end{bmatrix} Z\Omega\end{aligned}$$

have the following properties

a) $\det\mathcal{A} \neq 0, \det\mathcal{B} \neq 0$.
b) *the numbers $\delta_{jk}, k = 1, 2; j = 0, 1, 2$ are not integers.*
c) *for the pair $\mathcal{A}$, $\mathcal{B}$ one of the following three conditions* 1), 2), 3) *is fulfilled:*
   1) *$\mathcal{A}$ and $\mathcal{B}$ have no common eigenvectors and $l_1 = -l_2$,*
   2) *$\mathcal{A}$ and $\mathcal{B}$ do not commute, posses a common eigenvector, and $l_1 = -l_2 \geq 0$,*
   3) *$\mathcal{A}$ and $\mathcal{B}$ commute and $l_1 = -l_2 = 0$.*

We would like to make some observations regarding the operators $A_{\mathbb{T}}$ and $B_{\mathbb{R}}$. In [4], formulas for the partial indices of piecewise constant matrix functions with three values on the real axis have been calculated. These results allow us to find the dimensions of the kernels of $A_{\mathbb{T}}$ and $B_{\mathbb{R}}$.

Knowing the factors of piecewise constant matrix functions with three values on the real axis provides an opportunity to obtain the solvability conditions and to effectively construct the general solution of the equation with the operators $A_{\mathbb{T}}$ and $B_{\mathbb{R}}$.

In the article [7] a Riemann boundary value problem with shift was considered. The problem was reduced to a special case of the operator $B_{\mathbb{R}}$.

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