SENSIVITY ANALYSIS OF THE REPLACEMENT PROBLEMCAPÍTULO: INGENIERÍA Y GESTIÓN DE SISTEMAS

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Abstract

The replacement problem can be modeled as a finite, irreducible, homogeneous Markov Chain. In our proposal we modeled the problem using a Markov decision process and then, the instance is optimized using linear programming.

Our goal is to analyze the sensitivity and robustness of the optimal solution across the perturbation of the optimal basis (B^*) which is obtained from the simplex algorithm in order to comprehend how the optimal solution changes with a slight change in the transition probabilities matrix . The perturbation ($\tilde{B}$) can be approximated by a given matrix H such that $\tilde{B} = kB + H$. Some algebraic relations between the optimal solution (B^*) and the solution of the perturbed instance (B^*) are obtained, this is our approach, to establish some perturbation bounds through theorems and propositions.

Keywords:

Markov chains; linear programming; replacement problem.

1. Introduction

Machine replacement problem has been studied by a lot of researchers and is also an important topic in operations research, industrial engineering and management science. Items which are under constant usage, need replacement at an appropriate time as the efficiency of the operating system using such items suffer a lot.

In the real-world the equipment replacement problem involves the selection of two or more machines of one or more types from a set of several possible alternative machines with different capacities and cost of purchase and operation. When the problem involves a single machine, it is common to find two well-defined forms of this; the quantity-based replacement, and the time-based replacement. In the quantity-based replacement model, a machine is replaced when an accumulated product of size q is produced. In this model, one has to determine the optimal production size q. While in a time-based replacement model, a machine is replaced in every period of T with a profit maximizing.

When the problem involves two or more machines this problem is named the parallel machine replacement problem, and the time-based replacement model consists of finding a minimum cost replacement policy for a finite population of economically interdependent machines.

A replacement policy is a specification of “keep” or “replace” actions, one for each period. Two simple examples are the policy of replacing the equipment every time period and the policy of keeping the first machine until the end of a period N . An optimal policy is a policy that achieves the smallest total net cost of ownership over the entire planning horizon and it has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard the state resulting from the first decision. In practice, the replacement problem can be easily addressed using dynamic programming and Markov decision processes.

The dynamic programming uses the following idea: The system is observed over a finite or infinite horizon split up into periods or stages. At each stage the system is observed and a decision or action concerning the system has to be made. The decision influences (deterministically or o stochastically) the state to be observed at the next stage, and depending on the state and the decision made, an immediate reward is gained. The expected total rewards from the present stage and the one of the following state is expressed by the functional equation. Optimal decisions depending on stage and state are determined backwards step by step as those maximizing the right hand side of the functional equation.

Howard (1960 combines the dynamic programming technique with the mathematically well established notion of a Markov chain, creating the new concept called the Markov Decision processes and developing the solution of infinite stage problems. The policy iteration method was created as an alternative to the stepwise backward contraction methods. The policy iteration was a result of the application of the Markov chain environment and it was an important contribution to the development of optimization techniques (Kristensen 1996).

In this document, we consider a stochastic machine replacement model. The system consists of a single machine and this is assumed to operate continuously and efficiently over N periods. In each period, the quality of the machine deteriorates due to its use, and therefore, it can be in any of the N states, denoted 1, 2, …, N . In our proposal we modeled the problem using a Markov decision process and then, the instance is optimized using linear programming. Our goal is to analyze the sensitivity and robustness of the optimal solution across the perturbation of the optimal basis from the simplex algorithm.

Specifically the methodology used in this work is to model the replacement problem through a Markov decision process, optimize the instance obtained using linear programming, and finally, analyzing the sensitivity and robustness of the solution obtained by the perturbation of the optimal basis from the simplex algorithm and obtain algebraic relations between the initial optimal solution and the solution of the perturbed instance. In a real problem this will help to know if there is any change in the optimal solution when there are changes in the transition probabilities matrix. This transition probabilities matrix is not known with certainty, knowing exactly how much can be altered without affecting the optimal solution could lead to making better decisions regarding the replacement problem.

In our proposal we assume that for each new machine it state can become worse or may stay unchanged, and that the transition probabilities p_{ij} are known, where

p_{ij} =P {next state will be j | current state is i } =0, if $j < i$

p_{ij} is the probability of passing on her next stage to state j given that the current was in the state i. Also be assumed that the state of the machine is known at the start of each period, and we must choose one of the following two options: a) Let the machine operate one more period in the state it currently is, b) Replace the machine by a new one, where every new machines for replacement are assumed to be identical.

2. Literature Review

There are several theoretical models for determining the optimal replacement policy.

The basic model considers maintenance cost and resale value, which have their standard behavior as per the same cost during earlier period and also partly having an exponential grown pattern as per passage of time. Similarly the scrap value for the item under usage can be considered to have a similar type of recurrent behavior.

In relation to stochastic models the available literature on discrete time maintenance models predominantly treats an equipment deterioration process as a Markov chain.

Sernik and Marcus (1991) obtained the optimal policy and its associated cost for the two-dimensional Markov replacement problem with partial observations. They demonstrated that in the infinite horizon, the optimal discounted cost function is piecewise linear, and also provide formulas for computing the cost and the policy. In (Sethi et al. 2000), the authors assume that the deterioration of the machine is not a discrete process but it can be modeled as a continuous time Markov process, therefore, the only way to improve the quality is by replacing the machine by one new. They derive some stability conditions of the system under a simple class of real-time scheduling/replacement policy.

Some models are approached to evaluate the inspection intervals for a phased deterioration monitored complex components in a system with severe down time costs using a Markov model (see Sherwin and Al-Najjar 1999, for example).

In (Lewis 1987) the problem is approached from the perspective of the reliability engineering developing replacement strategies based on predictive maintenance. Moreover in (Childress and Durango-Cohen 1999) the authors formulated a stochastic version of the parallel machine replacement problem. They analyzed the structure of optimal policies under general classes of replacement cost functions.

Another important approach that has received the problem is the geometric programming (Cheng 1999). In its proposal, the author discusses the application of this technique to solving replacement problem with an infinite horizon and under certain circumstances he obtains a closed-form solution to the optimization problem.

A treatment to the problem when there are budget constraints can be found in (Karabakal et al., 2000). In their work, the authors propose a dual heuristic for dealing with large, realistically sized problems through the initial relaxation of budget constraints.

Compared with simulation techniques, Dohi et al. (2004), propose a technique based on obtaining the first two moments of the discounted cost distribution, and then, they approximate the underlying distribution function by three theoretical distributions using Monte Carlo simulation

The most important pioneers in applying dynamic programming models in replacement problems are: Bellman (1955), White (1969), Davidson (1970), Walker (1992) and Bertsekas (2000). Recently the Markov decision process has been applied successfully to the animal replacement problem as a productive unit (see Plà et al. 2004, Nielsen and Kristensen 2006, Nielsen et al. 2009, for example).

Although the modeling and optimization of the replacement problem using Markov decision processes is a topic widely known (Hillier and Lieberman 2002). However, there is a significant amount about the theory of stochastic perturbation matrices (see Schrijner and Doorn 2009, Meyer 1994, Abbad et al. 1990, Feinberg 2000, and references therein).

In literature there are hardly any results concerning the perturbation and robustness of the optimal solution of a replacement problem modeled via a Markov decision process and optimized using linear programming. In this paper we are interested in addressing this issue with a stochastic perspective.

3. Problem formulation

We start by defining a discrete-time Markov decision process with a finite state space Z states $z_1, z_2, ..., z_Z$ where, in each stage $s = 1,2,...$ the analyst should made a decision d between ξ possible. Denote by $z(n) = z$ and $d(n) = d_i$ the state and the decision made in stage

n respectively, then, the system moves at the next stage $n+1$ in to the next state j with a know probability given by

$$p_{zj}^k = \mathbf{P}\ \left[z(n+1) = j \ \middle|\ z(n) = z, d_n = d_k \right]$$

When the transition occurs, it is followed by the reward r_{zj}^k and the payoff is given by $\psi_z^k = \sum_{j=1}^{z} p_{zj}^k r_{zj}^k$ at the state z after the decision d_k is made.

For every policy $\vartheta(k_1, k_2, ..., k_Z)$, the corresponding Markov chain is ergodic, then the steady state probabilities of this chain are given by $p_z^\vartheta = \lim_{n\to\infty} P[z(n) = z], i = 1,2,..., Z$ and the problem is to find a policy ϑ for which the expected payoff

$$\Omega^\vartheta = \sum_{z=1}^{Z} p_z^\vartheta \psi_z^k \quad (1)$$

is maximum. In this system, the time interval between two transitions is called a stage.

An optimal policy is defined as a policy that maximizes (or minimizes) some predefined objective function. The optimization technique (i.e. the method to obtain an optimal policy) depends on the form of the objective function and it can result in different alternative objective function. The choice of criterion depends on whether the planning horizon is finite or infinite (Kristensen 1996).

In our proposal we consider a single machine and regular times intervals whether it should be kept for an additional period or it should be replaced by a new. By the above, the state space is defined by Z = {Keep (z_1), Replace (z_2)}, and having observed the state, action should be taken concerning the machine about to keep it for at least an additional stage or to replace it at the end of the stage.

The economic returns from the system will depend on its evolution and whether the machine is kept or replaced, in this proposal this is represented by a reward depending on state and action specified in advance. If the action replace is taken, we assume that the replacement takes place at the end of the stage at a known cost, the planning horizon is unknown and it is regarded infinite, also, all the stages are of equal length.

The optimal criterion used in this document is the maximization of the expected average reward per unit of time given by

$$h(\vartheta) = \sum_{z=1}^{Z} \pi_i^{\vartheta} r_i^{\vartheta}, \qquad (2)$$

where π_i^{ϑ} is the limiting state probability under the policy ϑ, and the optimization technique used is the linear programming (LP). Thus, we may maximize the problem (1) using the equivalent linear programming (Ross 1992) given by

$$\left.\begin{array}{l} \text{Maximize} \quad R = \sum_{z=1}^{Z} \sum_{k=1}^{\xi} r_{zk}\, x_{zk}, \\ \text{Subject to} \\ \sum_{z=1}^{Z} \sum_{k=1}^{\xi} x_{zk} = 1 \\ \sum_{k=1}^{\xi} x_{jk} - \sum_{z=1}^{Z} \sum_{k=1}^{\xi} x_{zk} p_{zj}(k) = 0, \text{para} \quad j = 1, 2, \ldots, Z. \\ x_{zk} \geq 0 \quad \text{for} \quad z = 1, 2, \ldots, Z \quad \text{and} \quad k = 1, 2, \ldots, \xi. \end{array}\right\} \qquad (3)$$

where x_z^k is the steady-state unconditional probability that the system is in state z and the decision k is made; similarly r_z^k is the reward obtained when the system is in state state z and the decision k is made. In this sense, k is optimal in state z if and only if, the optimal solution of (2) satisfy the unconditional probabilities x_z^k that the system visit the state Z, when making the decision k are strictly positive. Note that, the optimal value of the objective function is equal to the average rewards per stage under an optimal policy. The optimal value of $\sum_{k=1}^{\xi} x_z^k$ is equal to the limiting state probability π_z under an optimal policy. Model (3) contains ($\xi + 2$) functional contraints and $k(\xi + 1)$ decision variables. In (Pérez et al. 2006) it was showed that problem (3) has a degenerate basic feasible solution. In the remainder of this document, we are interested in the optimal basis associated with the solution of the problem (3) when it is solved via the simplex method.

4. Properties of the perturbed optimal basis associated with the replacement problem.

Without generality, a LP model (3) that optimizes a Markov chain can be defined as:

$$\left.\begin{array}{l}\text{Minimize } f(x) = c^t\,x \\ \text{subject to} \\ Ax = b, x \geq 0,\ A_{m\times n},\ c,\ x, \in\ \mathbb{R}^n, b \in \mathbb{R}^m\end{array}\right\} \tag{4}$$

In the LP model (4), the number of basic solutions ρ is less than or equal to the number of combinations $C(n,m)$ and B_{mxm} (submatrix of A) is a feasible basis of the LP model $B \in S$ that satisfies $S = \{B_i \in A : B^{-1}b \geq 0\}$.

Let $B^* \in S$ the optimal basis associated to the problem (4), and $\tilde{B}^*$ the perturbed matrix of B^*, defined by $\tilde{B}^* = kB^* + H$ where $k = 1$ and H is a matrix with the same order than B^*. The optimal solution is $x^* = (B^*)^{-1}b$ and any feasible solution is $\tilde{x} = (\tilde{B}^*)^{-1}b$. From these assumptions we state and prove the next propositions and theorems.

Proposition 4.1: $Let - dx = (x^* - \tilde{x})$,

$$\begin{array}{l} (1)\ \ -dx = \left[(B^*)^{-1} - \tilde{B}^{-1}\right]\, b \\ (2)\ \ \tilde{f} = f^* - c^t\left[(B^*)^{-1} - \tilde{B}^{-1}\right]\, b \end{array}$$

$$where\ f^* = f(x^*) \leftarrow Min$$

Proof: By the definition of $\tilde{B}^* = kB^* + H$

$$\tilde{x} = \tilde{B}^{-1}b = [k\,B^*\ +\ H]^{-1}\, b \tag{5}$$

so

$$-dx = (x^* - \tilde{x}) = (B^*)^{-1}\, b\ -\ [k\,B^*\ +\ H]^{-1}\, b = \left[(B^*)^{-1} - \tilde{B}^{-1}\right]\, b \tag{6}$$

Similarly

$$f(\tilde{x}) = f(x^* + dx) = c^t(x^* + dx) = f^* + c^t dx = f^* - c^t \left[(B^*)^{-1} - \tilde{B}^{-1} \right] b \qquad (7)$$

□

Proposition 4.2: *The matrix* H *is defined by*

$$H = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ h_{21} & h_{22} & \cdots & h_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ h_{m1} & h_{m2} & \cdots & h_{mn} \end{pmatrix} = [H_1, H_2, \ldots, H_n] \qquad (8)$$

where h_{ij} are the entries of H that could be perturbed.

The columns of the optimal basis B^* and the perturbed basis $\tilde{B}^*$ must sum 1.

$$\begin{aligned} \mathbf{1}\, B_j^* &= 1 \\ \mathbf{1}\, \tilde{B}_j &= 1, \end{aligned} \qquad (9)$$

Proof: The proof is trivial. The optimal basis is composed by the transitions probabilities matrix of P, considering the properties of the Markov chain we have

$$\pi_j = \sum_{z=0}^{Z} \pi_z p_{zj}, \quad \forall \;\; j = 0, 1, \ldots, Z \qquad (10)$$

where $\pi_j = \lim_{n\to\infty} p_{zj}^n$, the equation (10) is defined by $\pi P^t = \pi$, then for is fulfilled that. This property is valid also for $\tilde{B}^*$ □

Theorem 4.3: The Euclidean norm is used to establish perturbation bounds between the optimal basis B^* and the perturbed basis $\tilde{B}^*$, such that

$$\| x^* - \tilde{x} \|_2 \leq \| (B^*)^{-1} - (\tilde{B}^*)^{-1} \|_2 \qquad (11)$$

Proof:

$$\begin{aligned} \| x^* - \tilde{x} \|_2 &= \| (B^*)^{-1} b \; - \; (\tilde{B}^*)^{-1} b \|_2 \\ &= \| b \left[(B^*)^{-1} \; - \; (\tilde{B}^*)^{-1} \right] \|_2 \\ &\leq \| b \|_2 \; \cdot \; \| (B^*)^{-1} \; - \; (\tilde{B}^*)^{-1} \|_2 \\ &= \| (B^*)^{-1} \; - \; (\tilde{B}^*)^{-1} \|_2 \end{aligned} \tag{12}$$

because $\| b \|_2 = 1$ □

Proposition 4.4:

$$\tilde{x} = (\tilde{B}^*)^{-1} \; B^* \; x^* \tag{13}$$

Proof: From the LP model (4)

$$x^* = (B^*)^{-1} \; b, \tag{14}$$

$$\tilde{x} = (\tilde{B}^*)^{-1} \; b, \tag{15}$$

premultiplying the equation (14) times B^* ,

$$\begin{aligned} &B^* \; x^* = B^* \; (B^*)^{-1} \; b, \quad \text{so} \\ &B^* \; x^* = b \end{aligned} \tag{16}$$

similarly, premultiplying the equation (15) times $\tilde{B}^*$,

$$\begin{aligned} &\tilde{B}^* \; \tilde{x} = \tilde{B}^* \; (\tilde{B}^*)^{-1} \; b, \quad \text{so} \\ &\tilde{B}^* \; \tilde{x} = b \end{aligned} \tag{17}$$

equalizing (16) and (17),

$$\tilde{B}^* \; \tilde{x} = B^* \; x^* \tag{18}$$

isolating B^* results the equation (13) □

Theorem 4.5: A feasible solution satisfies that $D_{i1} \geq 0, i = 1,2,...,n$ where $D = (B^* + H)^{-1}$.

$$\| x^* - \tilde{x} \|_2 \leq \| (B^*)^{-1} - (\tilde{B}^*)^{-1} \|_2 \quad (11)$$

Proof: Let $\widetilde{B}* = kB* + H$ and $\tilde{x} = (\widetilde{B}*)^{-1} b \geq 0$, then for $k = 1$

$$\tilde{x} = (B^* + H)^{-1} \cdot b,$$

$$= D \cdot b = \begin{bmatrix} D_{11} & D_{12} & \ldots D_{1m} \\ D_{21} & D_{22} & \ldots D_{2m} \\ \ldots & \ldots & \ldots\ldots \\ D_{m1} & D_{m2} & \ldots D_{mm} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ \ldots \\ 0 \end{bmatrix} = \begin{bmatrix} D_{11} \geq 0 \\ D_{21} \geq 0 \\ \ldots \\ D_{m1} \geq 0 \end{bmatrix} \quad (19)$$

□

5. Numerical example.

Consider the following transition probabilities reported in (Kristensen 1996), which represented a Markovian decision process with $d = \{K, R\}$

Table 1. Transition probabilities

p_{zj}^d*	d = 1 (Keep)			d = 2 (Replace)		
	j=1(B)	j=2(P)	j=3(A)	j=1(B)	j=2(P)	j=3(A)
z=1	0.6000	0.3000	0.1000	0.3333	0.3333	0.3333
z=2	0.2000	0.6000	0.2000	0.3333	0.3333	0.3333
z=3	0.1000	0.3000	0.6000	0.3333	0.3333	0.3333

* p_{zj}^d is the probability of going from state z to state j considering the k decision

Therefore the transition probabilities matrices are:

$$K = \begin{bmatrix} 0.6\ 0.3\ 0.1 \\ 0.2\ 0.6\ 0.2 \\ 0.1\ 0.3\ 0.6 \end{bmatrix} \quad R = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \tag{20}$$

In order to maximize the objective function the cost coefficients are

Table 2. Cost coefficients

r^d_{zj}*	d = 1 (Keep)	d = 2 (Replace)
z=1	10,000	9,000
z=2	12,000	11,000
z=3	14,000	13,000

* r^d_{zj} is the earning of going from state z to state j considering the k decision

The corresponding LP problem is:

$$\left.\begin{array}{l}
\text{Maximize} \\
R = 10,000x_{11} + 9,000x_{12} + 12,000x_{21} + 11,000x_{22} + 14,000x_{31} + 13000x_{32} \\
\text{subject to} \\
x_{11} + x_{12} + x_{21} + x_{22} + x_{31} + x_{32} = 1 \\
\frac{2}{5}x_{11} + \frac{2}{3}x_{12} - \frac{1}{5}x_{21} - \frac{1}{3}x_{22} - \frac{1}{10}x_{31} - \frac{1}{3}x_{32} = 0 \\
\frac{-3}{10}x_{11} - \frac{1}{3}x_{12} + \frac{2}{5}x_{21} + \frac{2}{3}x_{22} - \frac{3}{10}x_{31} - \frac{1}{3}x_{32} = 0 \\
\frac{-1}{10}x11 - \frac{1}{3}x_{12} - \frac{1}{5}x_{21} - \frac{1}{3}x_{22} + \frac{2}{5}x_{31} - \frac{2}{3}x_{32} = 0 \\
x_{ij} \ \geq \ 0 \ \ \forall \ i, j
\end{array}\right\} \tag{21}$$

The optimal inverse basis $(B^*)^{-1}$ of the LP problem associated to this solution is:

$$(B^*)^{-1} = \begin{pmatrix} \frac{3}{16} & 0 & -\frac{9}{8} & -\frac{21}{16} \\ 0 & 1 & 1 & 1 \\ \frac{7}{16} & 0 & \frac{11}{8} & -\frac{1}{16} \\ \frac{3}{8} & 0 & -\frac{1}{4} & \frac{11}{8} \end{pmatrix} \tag{22}$$

The optimal solution and the basic variables of the inverse basis (are presented in order): $X_B = (x_{12}, a_2, x_{21}, x_{31}) = (0.1875,0,0.4375,0.375)$. The optimal value of the objective function is $12{,}187.50$. The basis $B*$ that will be perturbed is formed by the columns $(x_{12}, a_2, x_{21}, x_{31})$

$$B^* = \begin{pmatrix} 1 & 0 & 1 & 1 \\ \frac{2}{3} & 1 & -\frac{1}{5} & -\frac{1}{10} \\ -\frac{1}{3} & 0 & \frac{2}{5} & -\frac{3}{10} \\ -\frac{1}{3} & 0 & -\frac{1}{5} & \frac{2}{5} \end{pmatrix} \quad (23)$$

Note that $B*$ satisfies the **Proposition 4.2** that corresponds with the equation (9), this property must be conserved for $\tilde{B}*$.

Suppose that we are interested to perturb x_{12}. This decision variable has associated the transition probability $p_{11}(2) = 1/3$. Simplifying the restriction of the state 1 in the LP model (21), the value for this variable is $x_{12} - \frac{1}{3}x_{12} = \frac{2}{3}x_{12}$.

Continuing with the process, the restrictions of the states 2 and 3 are respectively:

$$-x_{12}p_{12}(2) = -\frac{1}{3}x_{12}, \quad -x_{12}p_{13}(2) = -\frac{1}{3}x_{12} \quad (24)$$

Because the restrictions of the LP model (3) the probability is affected by a minus sign. In $\tilde{B}*$, the variable x_{12} is associated with the vector $(1,2/3,-1/3,-1/3)^t$, and the positions that could be perturbed $2/3,-1/3,-1/3$ considering the equation (9). Note that the first element of the vector does not have any perturbation, because it corresponds to the first restriction of the LP model (3).

Suppose also, that the column vector $(1,2/3,-1/3,-1/3)^t$ of the matrix $\tilde{B}*$ that corresponds to the variable x_{12} will be perturbed in the second position, from $\frac{2}{3}$ to $(\frac{2}{3} + \in)$, $\in \neq 0$. The perturbed vector is

$$\left(1, \frac{2}{3} + \epsilon, \frac{-1}{3} - \frac{\epsilon}{2}, \frac{-1}{3} - \frac{\epsilon}{2}\right) \quad (25)$$

So the H matrix is:

$$H = \begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{pmatrix} = \begin{pmatrix} 0 & 0\,0\,0 \\ \epsilon & 0\,0\,0 \\ -\epsilon/2 & 0\,0\,0 \\ -\epsilon/2 & 0\,0\,0 \end{pmatrix} \qquad (26)$$

therefore the perturbed matrix is

$$\tilde{B}^* = \begin{pmatrix} 1 & 0 & 1 & 1 \\ \mathbf{\frac{2}{3}} + \epsilon & 1 & -\frac{1}{5} & -\frac{1}{10} \\ -\mathbf{\frac{1}{3}} - \mathbf{\frac{\epsilon}{2}} & 0 & \frac{2}{5} & -\frac{3}{10} \\ -\mathbf{\frac{1}{3}} - \mathbf{\frac{\epsilon}{2}} & 0 & -\frac{1}{5} & \frac{2}{5} \end{pmatrix} \qquad (27)$$

Every value of $H_1 = (h_{11}, h_{21}, h_{31}, h_{41})^t = (0, \in, -\frac{\in}{2}, -\frac{\in}{2})^t$ is associated with the decision $k = 2$ (replace) and the state $z = 1$ (the variable associated with this column vector is $x_{zk} = x_{12}$), because of this, any perturbation in H_1 will affect the R matrix in the first column

The R matrix is now

$$R = \begin{bmatrix} \frac{1}{3} - \epsilon & \frac{1}{3} + \frac{\epsilon}{2} & \frac{1}{3} + \frac{\epsilon}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \qquad (28)$$

The K matrix has not changes

Considering the equation (19) of the **Theorem 4.5**, $\tilde{x}$ is obtained,

$$\tilde{x} = (B^* + H)^{-1} \cdot b$$

$$= \begin{bmatrix} 1 & 0 & 1 & 1 \\ \mathbf{\frac{2}{3}} + \epsilon & 1 & -\frac{1}{5} & -\frac{1}{10} \\ -\mathbf{\frac{1}{3}} - \frac{\epsilon}{\mathbf{2}} & 0 & \frac{2}{5} & -\frac{3}{10} \\ -\mathbf{\frac{1}{3}} - \frac{\epsilon}{\mathbf{2}} & 0 & -\frac{1}{5} & \frac{2}{5} \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{10\left(\frac{8}{15}+\frac{13e}{20}\right)} & 0 & -\frac{3}{5\left(\frac{8}{15}+\frac{13e}{20}\right)} & -\frac{7}{10\left(\frac{8}{15}+\frac{13e}{20}\right)} \\ 0 & 1 & 1 & 1 \\ \frac{\frac{7}{30}+\frac{7e}{20}}{\frac{8}{15}+\frac{13e}{20}} & 0 & \frac{\frac{11}{15}+\frac{e}{2}}{\frac{8}{15}+\frac{13e}{20}} & \frac{-\frac{1}{30}-\frac{e}{2}}{\frac{8}{15}+\frac{13e}{20}} \\ \frac{\frac{1}{5}+\frac{3e}{10}}{\frac{8}{15}+\frac{13e}{20}} & 0 & \frac{-\frac{2}{15}-\frac{e}{2}}{\frac{8}{15}+\frac{13e}{20}} & \frac{\frac{11}{15}+\frac{e}{2}}{\frac{8}{15}+\frac{13e}{20}} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{29}$$

$$= \begin{bmatrix} \frac{1}{10\left(\frac{8}{15}+\frac{13e}{20}\right)} \geq 0 \\ 0 = 0 \\ \frac{\frac{7}{30}+\frac{7e}{20}}{\frac{8}{15}+\frac{13e}{20}} \geq 0 \\ \frac{\frac{1}{5}+\frac{3e}{10}}{\frac{8}{15}+\frac{13e}{20}} \geq 0 \end{bmatrix}$$

Solving the inequality associated with the first element $\frac{1}{10(8/15+13\in/20)} \geq 0$, an interval $(-32/39,\infty)$ is obtained. The second element fulfills with the equality. The third element have an inequality $\frac{7/30+7\in/20}{8/15+13\in/20} \geq 0$, the solution is $(-\infty,-32/39) \cup [-2/3,\infty)$. In the inequality $\frac{1/5+3\in/10}{8/15+13\in/20} \geq 0$, the solution interval is $(-2/3,\infty)$. The intersection of the intervals is $(-2/3,\infty)$, considering that the probabilities are between 0 and 1, the extent to perturb $\in$ in this particular case is $(-2/3,1]$ to conserve the feasibility of the perturbed solution $\tilde{x}$. Considering this perturbation interval we have that

1. Numerical comparative of equation (6), **Proposition 4.1** (1). The second column of the Tables 3 and 4 of the Appends is obtained directly from de LP model doing the perturbation of epsilon presented in the first column. The third column is found by

subtracting the optimal solution of the perturbed solution. The fourth column is obtained doing the matrix operations to demonstrate that the third and fourth columns are equal.

2. Also the numerical comparative of equation (6), **Proposition 4.1** (2) is presented in the Tables 5 and 6 of the Appends. The second and third columns of the tables are obtained directly from de LP model doing the perturbation of epsilon of the first column. The fourth column is obtained doing the matrix operations to demonstrate that the third and fourth columns are equal.
3. Note in Tables 7 and 8 that the second column is obtained with the LP model doing the perturbation showed in the first column. The third column is obtained doing the matrix operations to demonstrate that the second and third columns are equal. This results corresponds to equation (13), **Proposition 4.1**
4. Finally the numerical comparative of equation (11), **Theorem 4.3** is presented in Tables 9 and 10 of the Appends. Again the first column presents the value of the perturbation, the second one shows the euclidean norm of the difference between optimal and perturbed solutions, and the third column presents the euclidean norm between the optimal and the perturbed basis. Note that the second column is always less than the third one. Figure 11 also shows this.

6. Conclusions and future work

In this document we considered a stochastic machine replacement problem with a single machine that operates continuously and efficiently over N periods, we were interested in the matrix perturbation procedure from a probabilistic point of view, because there is no assurance that the probabilities of transition matrices from one state to another could change and with this, make the solution and the decision problem associated with the replacement problem could change also. Additionally, a perturbation could cause structural changes in the probability transitions matrices, causing that two states that were originally communicated, now do not, and thus also affect the decision.

The replacement problem was solved for different authors using dynamic and linear programming. However, the perturbation associated with the transition probability matrices is a recent topic (Pérez, 2006), with a lot to explore. The original contribution in this work is perturbed the optimal basis $B*$, demonstrated that the perturbation in this optimal basis affected the transition probability matrices (*K, M)*, found that a feasibility region of perturbation exist, finally, that the optimal basis $B*$, the perturbed basis $\tilde{B}*$, the optimal solution $x*$ and the perturbed solutions $\tilde{x}$ are related. These results are obtained observed by experimentations and then demonstrate them mathematically.

The algebraic relations obtained, is proved in a numerical example of the literature, also are proved when the perturbation of the optimal basis is done is several elements of the matrix at once. We conclude that is possible to establish perturbation bounds between the optimal solution $x*$ and the perturbed solution $\tilde{x}$ across the perturbation of the optimal basis $B*$.

Future work could be consider other perturbations over the optimal basis $B*$ (in this document the perturbation used is $\tilde{B}* = kB* + H$) and perturb the entries of the matrix as random variables, because because it would be interesting to evaluate how the optimal solution and the decision change with other types of perturbations.

REFERENCES

[1] Abbad, M., Filar, J., and Bielecki, T.R., 1990. Algorithms for singularly perturbed Markov chain. *In: Proceedings of the 29th IEEE Conference*, Honolulu, HI, USA, 5-7 Dec 1990. Decision and Control, 1402-1407.

[2] Bellman, R., 1955. Equipment Replacement Policy. *Journal of society for Industrial and Applied Mathematics*, 3, 133-136.

[3] Bertsekas, D., 2000. *Dynamic programming and optimal control*. Athena Scientific, Belmont, Massachusetts.

[4] Cheng, T., 1999. Optimal replacement of ageing equipment using geometric programming. *International Journal of Production Research*, 30 (9), 2151-2158.

[5] Childress, S. and Durango-Cohen, P., 1999. On parallel machine replacement problems with General replacement cost functions and stochastic deterioration. *Naval Research Logistics*, 52, 402-419.

[6] Davidson, D. 1970. An Overhaul Policy for Deteriorating Equipment. *In: A. Jardine, ed. Operational Research in Maintenance*. New York, NY: Manchester University Press, chap. 6, 1-24.

[7] Dohi, T., et al., 2004. A simulation study on the discounted cost distribution under age replacement policy. *IEMS*, 3 (2), 134-139.

[8] Feinberg, E., 2000. Constrained discounted decision processes and Hamiltonian cycles. *Mathematical Operational Research*, 25, 130-140.

[9] Hillier, F., and Lieberman, G., 2002. *Introduction to Operations Research.* Mc. Graw Hill Companies, Inc.

[10] Howard, R.A., 1960. *Dynamic programming and Markov Processes*. Cambridge Massachusetts:The MIT Press.

[11] Karabakal, N., Bean, C., and Lohman, J., 2000. Solving large replacement problems with budget Constraints. *The engineering economist*, 4(5), 290-308.

[12] Kristensen, A., Textbook notes of herd management: Dynamic programming and Markov process. 1996., Dina notat 49 Available from http://www.husdyr.kvl.dk/ark/pdf/notat49.pdf.

[13] Lewis, E., 1987. *Introduction to Reliability Theory*. Wiley, Singapore.

[14] Meyer, C., 1994. Sensitivity of the stationary distribution of a Markov chain. *SIAM Journal of Matrix Applications*, 15, 715-728.

[15] Nielsen, L., et al., 2009. Optimal replacement policies for dairy cows based on daily yield measurements. *Journal of Dairy Science*, 93, 75-92.

[16] Nielsen, L. and Kristensen, A., 2006. Finding the K best policies in a finite-horizon Markov decision process.*European Journal of Operation Research*, 175, 1164-1179.

[17] Pèrez, G., Àlvarez, M., and Garnica, J., 2006. Stochastic Linear Programming to Optimize some Stochastic Systems. *WSEAS Transaction of Systems*, 9, 2263-2267.

[18] Plà, L., Pomar C., and Pomar J., 2004. A Markow sow model representing the productive lifespan of herd sows. *Agricultural Systems*, 76, 253-272.

[19] Ross, S., 1992. *Applied probability models with optimization applications*. Sn Fco. Cal. Holdenday.

[20] Schrijner, P. and Doorn, E., 2009. The deviation matrix of a continuos-time Markov chain. *Probability in the engineering and informational science*, 16, 351-366.

[21] Sernik, E.L. and Marcus, S., 1991. Optimal cost and policy for markovian replacement problem. *Journal of optimization theory and applications*, 71 (1), 105-126.

[22] Sethi, S.P., Sorger, G., and Zhou, X., 2000. Stability of real-time lot scheduling and machine replacement policies with quality levels. *IEEE transactions on automatic control*, 45 (11), 2193-2196.

[23] Sherwin, J. D. and Al-Najjar, B., 1999. Practical models for conditions-based monitoring inspection intervals. *Journal of Quality in Maintenance Engineering*, 5, 203-209.
[24] Walker, J., 1992. Graphical Analysis for Machine Replacement. International *Journal of Operations and Production Management*, 14, 54-63.
[25] White, D., 1969. *Dynamic programming*. Sn Fco. Cal. Holden Day.

APPENDS

Table 3. Ascending perturbation in ϵ

ϵ	$\tilde{x}^a$	$x^* - \tilde{x}$	$\left[(B^*)^{-1} - (\tilde{B}^*)^{-1}\right] b^b$
0.01	(0.1852, 0, 0.4387, 0.3760)	(0.0023, 0, −0.0012, −0.0010)	(0.0023, 0, −0.0012, −0.0010)
0.02	(0.1830, 0, 0.4399, 0.3770)	(0.0045, 0, −0.0024, −0.0021)	(0.0045, 0, −0.0024, −0.0021)
0.03	(0.1808, 0, 0.4410, 0.3780)	(0.0066, 0, −0.0036, −0.0031)	(0.0066, 0, −0.0036, −0.0031)
0.04	(0.1787, 0, 0.4421, 0.3790)	(0.0087, 0, −0.0047, −0.0040)	(0.0087, 0, −0.0047, −0.0040)
0.05	(0.1763, 0, 0.4432, 0.3799)	(0.0108, 0, −0.0058, −0.0050)	(0.0108, 0, −0.0058, −0.0050)
0.06	(0.1747, 0, 0.4443, 0.3809)	(0.0128, 0, −0.0069, −0.0059)	(0.0128, 0, −0.0069, −0.0059)
0.07	(0.1727, 0, 0.4454, 0.3818)	(0.0147, 0, −0.0079, −0.0068)	(0.0147, 0, −0.0079, −0.0068)
0.08	(0.1708, 0, 0.4464, 0.3826)	(0.0167, 0, −0.0090, −0.0077)	(0.0167, 0, −0.0090, −0.0077)
0.09	(0.1689, 0, 0.4474, 0.3835)	(0.0185, 0, −0.0100, −0.0086)	(0.0185, 0, −0.0100, −0.0086)
0.1	(0.1671, 0, 0.4484, 0.3844)	(0.0204, 0, −0.0110, −0.0094)	(0.0204, 0, −0.0110, −0.0094)
0.2	(0.1508, 0, 0.4573, 0.3920)	(0.0367, 0, −0.0198, −0.0170)	(0.0367, 0, −0.0198, −0.0170)
0.3	(0.1373, 0, 0.4645, 0.3982)	(0.0502, 0, −0.0270, −0.0232)	(0.0502, 0, −0.0270, −0.0232)
0.4	(0.1261, 0, 0.4706, 0.4034)	(0.0614, 0, −0.0331, −0.0284)	(0.0614, 0, −0.0331, −0.0284)
0.5	(0.1165, 0, 0.4706, 0.4034)	(0.0710, 0, −0.0382, −0.0328)	(0.0710, 0, −0.0382, −0.0328)
0.6	(0.1083, 0, 0.4706, 0.4034)	(0.0792, 0, −0.0426, −0.0366)	(0.0792, 0, −0.0426, −0.0366)
0.7	(0.1012, 0, 0.4840, 0.4148)	(0.0863, 0, −0.0465, −0.0398)	(0.0863, 0, −0.0465, −0.0398)
0.8	(0.0949, 0, 0.4873, 0.4177)	(0.0926, 0, −0.0498, −0.0427)	(0.0926, 0, −0.0498, −0.0427)
0.9	(0.0894, 0, 0.4903, 0.4203)	(0.0981, 0, −0.0528, −0.0453)	(0.0981, 0, −0.0528, −0.0453)
1	(0.0845, 0, 0.4930, 0.4225)	(0.1030, 0, −0.0555, −0.0475)	(0.1030, 0, −0.0555, −0.0475)

[a] This value is obtained directly from the LP model.

[b] This value is obtained doing the matrix operations.

Table 4. Descending perturbation in ϵ

ϵ	$\tilde{x}^a$	$x^* - \tilde{x}$	$\left[(B^*)^{-1} - (\tilde{B}^*)^{-1}\right] b^b$
−0.01	(0.1898, 0, 0.4362, 0.3739)	(−0.0023, 0, 0.0012, 0.0011)	(−0.0023, 0, 0.0012, 0.0011)
−0.02	(0.1921, 0, 0.4349, 0.3728)	(−0.0047, 0, 0.0025, 0.0022)	(−0.0047, 0, 0.0025, 0.0022)
−0.03	(0.1946, 0, 0.4336, 0.3717)	(−0.0071, 0, 0.0038, 0.0033)	(−0.0071, 0, 0.0038, 0.0033)
−0.04	(0.1971, 0, 0.4323, 0.3705)	(−0.0096, 0, 0.0052, 0.0044)	(−0.0096, 0, 0.0052, 0.0044)
−0.05	(0.1996, 0, 0.4309, 0.3693)	(−0.0122, 0, 0.0066, 0.0056)	(−0.0122, 0, 0.0066, 0.0056)
−0.06	(0.2022, 0, 0.4295, 0.3681)	(−0.0148, 0, 0.0080, 0.0068)	(−0.0148, 0, 0.0080, 0.0068)
−0.07	(0.2049, 0, 0.4280, 0.3669)	(−0.0175, 0, 0.0094, 0.0081)	(−0.0175, 0, 0.0094, 0.0081)
−0.08	(0.2077, 0, 0.4265, 0.3656)	(−0.0203, 0, 0.0109, 0.0093)	(−0.0203, 0, 0.0109, 0.0093)
−0.09	(0.2106, 0, 0.4250, 0.3643)	(−0.0231, 0, 0.0124, 0.0107)	(−0.0231, 0, 0.0124, 0.0107)
−0.10	(0.2135, 0, .04234, 0.3629)	(−0.0260, 0, 0.0140, 0.0120)	(−0.0260, 0, 0.0140, 0.0120)
−0.20	(0.2979, 0, 0.4049, 0.3471)	(−0.0604, 0, 0.0325, 0.0279)	(−0.0604, 0, 0.0325, 0.0279)
−0.30	(0.2955, 0, 0.3793, 0.3251)	(−0.1081, 0, 0.0582, 0.0499)	(−0.1081, 0, 0.0582, 0.0499)
−0.40	(0.3658, 0, 0.3414, 0.2926)	(−0.1784, 0, 0.0960, 0.0823)	(−0.1784, 0, 0.0960, 0.0823)
−0.50	(0.4800, 0, 0.2800, 0.2400)	(−0.2925, 0, 0.1575, 0.1350)	(−0.2925, 0, 0.1575, 0.1350)
−0.60	(0.6976, 0, 0.1627, 0.1395)	(−0.5102, 0, 0.2747, 0.2355)	(−0.5102, 0, 0.2747, 0.2355)

[a] This value is obtained directly from the LP model.

[b] This value is obtained doing the matrix operations.

Table 5. Ascending perturbation in ϵ

ϵ	$\tilde{x}$	$f(\tilde{x})$[a]	$f^* - c^t\left[(B^*)^{-1} - (\tilde{B}^*)^{-1}\right] b$[b]
0.01	(0.1852, 0, 0.4387, 0.3760)	12, 196.4	12, 196.4
0.02	(0.1830, 0, 0.4399, 0.3770)	12, 205.0	12, 205.0
0.03	(0.1808, 0, 0.4410, 0.3780)	12, 213.4	12, 213.4
0.04	(0.1787, 0, 0.4421, 0.3790)	12, 221.7	12, 221.7
0.05	(0.1763, 0, 0.4432, 0.3799)	12, 299.7	12, 299.7
0.06	(0.1747, 0, 0.4443, 0.3809)	12, 237.6	12, 237.6
0.07	(0.1727, 0, 0.4454, 0.3818)	12, 245.3	12, 245.3
0.08	(0.1708, 0, 0.4464, 3826)	12, 252.8	12, 252.87
0.09	(0.1689, 0, 0.4474, 0.3835)	12, 260.2	12, 260.2
0.10	(0.1671, 0, 0.4484, 0.3844)	12, 267.4	12, 267.4
0.20	(0.1507, 0, 0.4572, 0.3919)	12, 231.7	12, 231.7
0.30	(0.1373, 0, 0.4645, 0.3918)	12, 384.4	12, 384.4
0.40	(0.1261, 0, 0.4706, 0.4034)	12, 429.0	12, 429.0
0.50	(0.1165, 0, 0.4706, 0.4034)	12, 466.0	12, 466.0
0.60	(0.1083, 0, 0.4706, 0.4034)	12, 498.0	12, 498.0
0.70	(0.1012, 0, 0.4840, 0.4148)	12, 526.0	12, 526.0
0.80	(0.0949, 0, 0.4873, 0.4177)	12, 551.0	12, 551.0
0.90	(0.0894, 0, 0.4903, 0.4203)	12, 572.0	12, 572.0
1	(0.0845, 0, 0.4930, 0.4225)	12, 592.0	12, 592.0

[a] This value is obtained directly from the LP model.

[b] This value is obtained doing the matrix operations.

Table 6. Descending perturbation in ϵ

ϵ	$\tilde{x}$	$f(\tilde{x})$[a]	$f^* - c^t\left[(B^*)^{-1} - (\tilde{B}^*)^{-1}\right] b$[b]
−0.01	(0.1898, 0, 0.4362, 0.3739)	12, 178.4	12, 178.4
−0.02	(0.1921, 0, 0.4349, 0.3728)	12, 169.1	12, 169.1
−0.03	(0.1946, 0, 0.4336, 0.3717)	12, 159.6	12, 159.6
−0.04	(0.1971, 0, 0.4323, 0.3705)	12, 149.8	12, 149.8
−0.05	(0.1996, 0, 0.4309, 0.3693)	12, 139.8	12, 139.8
−0.06	(0.2022, 0, 0.4295, 0.3681)	12, 129.5	12, 139.8
−0.07	(0.2049, 0, 0.4280, 0.3669)	12, 118.5	12, 118.5
−0.08	(0.2077, 0, 0.4265, 0.3656)	12, 108.0	12, 108.0
−0.09	(0.2106, 0, 0.4250, 0.3643)	12, 096.9	12, 096.9
−0.10	(0.2135, 0, 0.4234, 0.3629)	12, 085.4	12, 085.4
−0.20	(0.2979, 0, 0.4049, 0.3471)	11, 950.4	11, 950.4
−0.30	(0.2955, 0, 0.3793, 0.3251)	11, 763.5	11, 763.5
−0.40	(0.3658, 0, 0.3414, 0.2926)	11, 487.8	11, 487.8
−0.50	(0.4800, 0, 0.2800, 0.2400)	11, 040.0	11, 040.0
−0.60	(0.6976, 0, 0.1627, 0.1395)	10, 186.0	10, 186.0

[a] This value is obtained directly from the LP model.

[b] This value is obtained doing the matrix operations.

Table 7. Ascending perturbation in ϵ

ϵ	$\tilde{x}$[a]	$(\breve{B}^*)^{-1}\ B^*\ x^{*}$[b]
0.01	(0.1852, 0, 0.4387, 0.3760)	(0.1852, 0, 0.4387, 0.3760)
0.02	(0.1830, 0, 0.4399, 0.3770)	(0.1830, 0, 0.4399, 0.3770)
0.03	(0.1808, 0, 0.4410, 0.3780)	(00808, 0, 0.4410, 0.3780)
0.04	(0.1787, 0, 0.4421, 0.3790)	(0.1787, 0, 0.4421, 0.3790)
0.05	(0.1763, 0, 0.4432, 0.3799)	(0.1763, 0, 0.4432, 0.3799
0.06	(0.1747, 0, 0.4443, 0.3809)	(0.1747, 0, 0.4443, 0.3809)
0.07	(0.1727, 0, 0.4454, 0.3818)	(0.1727, 0, 0.4454, 0.3818)
0.08	(0.1708, 0, 0.4464, 0.3826)	(0.1708, 0, 0.4464, 0.3826)
0.09	(0.1689, 0, 0.4474, 0.3835)	(0.1689, 0, 0.4474, 0.3835)
0.10	(0.1671, 0, 0.4484, 0.3844)	(0.1671, 0, 0.4484, 0.3844)
0.20	(0.1508, 0, 0.4573, 0.3920)	(0.1508, 0, 0.4573, 0.3920)
0.30	(0.1373, 0, 0.4645, 0.3982)	(0.1373, 0, 0.4645, 0.3982)
0.40	(0.1261, 0, 0.4706, 0.4034)	(0.1261, 0, 0.4706, 0.4034)
0.50	(0.1165, 0, 0.4706, 0.4034)	(0.1165, 0, 0.4706, 0.4034)
0.60	(0.1083, 0, 0.4706, 0.4034)	(0.1083, 0, 0.4706, 0.4034)
0.70	(0.1012, 0, 0.4840, 0.4148)	(0.1012, 0, 0.4840, 0.4148)
0.80	(0.0949, 0, 0.4873, 0.4177)	(0.0949, 0, 0.4873, 0.4177)
0.90	(0.0894, 0, 0.4903, 0.4203)	(0.0894, 0, 0.4903, 0.4203)
1	(0.0845, 0, 0.4930, 0.4225)	(0.0845, 0, 0.4930, 0.4225)

[a] This value is obtained directly from the LP model.

[b] This value is obtained doing the matrix operations.

Table 8. Descending perturbation in ϵ

ϵ	$\tilde{x}$[a]	$(\breve{B}^*)^{-1}\ B^*\ x^{*}$[b]
−0.01	(0.1898, 0, 0.4362, 0.3739)	(0.1898, 0, 0.4362, 0.3739)
−0.02	(0.1921, 0, 0.4349, 0.3728)	(0.1921, 0, 0.4349, 0.3728)
−0.03	(0.1946, 0, 0.4336, 0.3717)	(0.1946, 0, 0.4336, 0.3717)
−0.04	(0.1971, 0, 0.4323, 0.3705)	(0.1971, 0, 0.4323, 0.3705)
−0.05	(0.1996, 0, 0.4309, 0.3693)	(0.1996, 0, 0.4309, 0.3693)
−0.06	(0.2022, 0, 0.4295, 0.3681)	(0.2022, 0, 0.4295, 0.3681)
−0.07	(0.2049, 0, 0.4280, 0.3669)	(0.2049, 0, 0.4280, 0.3669)
−0.08	(0.2077, 0, 0.4265, 0.3656)	(0.2077, 0, 0.4265, 0.3656)
−0.09	(0.2106, 0, 0.4250, 0.3643)	(0.2106, 0, 0.4250, 0.3643)
−0.10	(0.2135, 0, 0.4234, 0.3629)	(0.2135, 0, 0.4234, 0.3629)
−0.20	(0.2979, 0, 0.4049, 0.3471)	(0.2979, 0, 0.4049, 0.3471)
−0.30	(0.2955, 0, 0.3793, 0.3251)	(0.2955, 0, 0.3793, 0.3251)
−0.40	(0.3658, 0, 0.3414, 0.2926)	(0.3658, 0, 0.3414, 0.2926)
−0.50	(0.4800, 0, 0.2800, 0.2400)	(0.4800, 0, 0.2800, 0.2400)
−0.60	(0.6976, 0, 0.1627, 0.1395)	(0.6976, 0, 0.1627, 0.1395)

[a] This value is obtained directly from the LP model.

[b] This value is obtained doing the matrix operations.

Table 9. Ascending perturbation in ϵ

ϵ	$\| x^* - \tilde{x} \|_2$	$\| (B^*)^{-1} - (\tilde{B}^*)^{-1} \|_2$
0.01	0.0028	0.0257
0.02	0.0055	0.0507
0.03	0.0081	0.0752
0.04	0.0107	0.0991
0.05	0.0132	0.1224
0.06	0.0157	0.1453
0.07	0.0181	0.1676
0.08	0.0204	0.1894
0.09	0.0227	0.2107
0.1	0.0250	0.2316
0.2	0.0450	0.4178
0.3	0.0615	0.5707
0.4	0.0753	0.6986
0.5	0.0870	0.8071
0.6	0.0971	0.9004
0.7	0.1058	0.9814
0.8	0.1135	1.0524
0.9	0.1202	1.1151
1	0.1263	1.1709

Table 10. Descending perturbation in ϵ

ϵ	$\| x^* - \tilde{x} \|_2$	$\| (B^*)^{-1} - (\tilde{B}^*)^{-1} \|_2$
−0.01	0.0028	0.0263
−0.02	0.0057	0.0533
−0.03	0.0087	0.0809
−0.04	0.0118	0.1092
−0.05	0.0149	0.1383
−0.06	0.0181	0.1682
−0.07	0.0214	0.1988
−0.08	0.0248	0.2303
−0.09	0.0283	0.2626
−0.1	0.0319	0.2959
−0.2	0.0741	0.6871
−0.3	0.1325	1.2286
−0.4	0.2187	2.0277
−0.5	0.3586	3.3254
−0.6	0.6254	5.8002

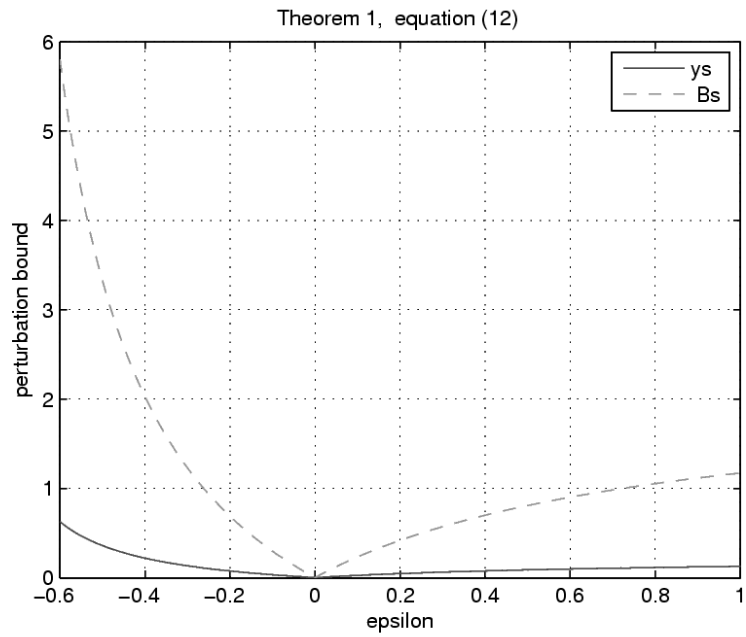

Figure 1. **Theorem 4.3**