REALIZATION OF A SIMPLE HIGHER DIMENSIONAL NONCOMMUTATIVE TORUS AS A TRANSFORMATION GROUP C*-ALGEBRA

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Abstract. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix, and let A_θ be the corresponding simple higher dimensional noncommutative torus. Suppose that d is odd, or that $d \geq 4$ and the entries of θ are not contained in a quadratic extension of $\mathbb{Q}$. Then A_θ is isomorphic to the transformation group C*-algebra obtained from a minimal homeomorphism of a compact connected one dimensional space locally homeomorphic to the product of the interval and the Cantor set. The proof uses classification theory of C*-algebras.

0. Introduction

Let θ be a skew symmetric real $d \times d$ matrix. Recall that the noncommutative torus A_θ is by definition [21] the universal C*-algebra generated by unitaries $u_1, u_2, \ldots, u_d$ subject to the relations

$$u_k u_j = \exp(2\pi i \theta_{j,k}) u_j u_k$$

for $1 \leq j, k \leq d$. (Of course, if all $\theta_{j,k}$ are integers, it is not really noncommutative. Also, some authors use $\theta_{k,j}$ in the commutation relation instead. See for example [9].) The algebras A_θ are natural generalizations of the rotation algebras to more generators. They, and their standard smooth subalgebras, have received considerable attention. As just a few examples, we mention [20], [3], [1], [22] and [4]. In [17] (also see the unpublished preprint [16]), is it proved that every simple higher dimensional noncommutative torus is an AT algebra.

In this paper, we prove that almost every simple higher dimensional ($d \geq 3$) noncommutative torus can be realized as the transformation group C*-algebra obtained from a minimal homeomorphism of a compact connected one dimensional space. The minimal homeomorphism is an irrational time map of the suspension flow of the restriction to its minimal set of a suitable Denjoy homeomorphism of the circle. The only exceptional cases are when d is even and there is a quadratic extension of $\mathbb{Q}$ which contains all the entries of θ. The proof consists of constructing a homeomorphism, of the type described, whose transformation group C*-algebra has the same Elliott invariant as A_θ, and using the classification results of [12], [13], and [17] (also see the unpublished preprint [16]).

In the first section, we prove the result under the assumption that the image of $K_0(A_\theta)$ under the unique tracial state of A_θ has rank at least 3. In Section 2, we prove that the rank can be 2 only when d is even and there is a quadratic extension

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of $\mathbb{Q}$ which contains all the entries of θ. In Section 3, we give a kind of converse result for the three dimensional case.

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1. Construction of the homeomorphisms

Denjoy homeomorphisms of the circle are described in Section 3 of [19]. In particular, if $h_0 \colon S^1 \to S^1$ is a Denjoy homeomorphisms of the circle with rotation number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, there is an associated set $Q(h_0) \subset S^1$ of "accessible points", defined up to a rigid rotation of the circle, as in Definition 3.5 of [19]. The homeomorphism h_0 has a unique minimal set X_0, which is homeomorphic to the Cantor set, and the set $Q(h_0)$ can be thought of as the set of points at which S^1 is "cut" to build $h_0|_{X_0}$ from the rotation R_α by α. In particular, $Q(h_0)$ consists of the points in S^1 which lie on a number n of orbits of R_α, with $1 \leq n \leq \infty$.

Definition 1.1. A *restricted Denjoy homeomorphism* is the restriction of a Denjoy homeomorphism h_0 to its unique minimal set. The restricted Denjoy homeomorphism is said to have *cut number* $n \in \{1, 2, \ldots, \infty\}$ if $Q(h_0)$ consists of exactly n orbits of the associated rotation on S^1.

The cut number n is called $n(h_0)$ in [19]. It depends only on the restriction of h_0 to its minimal set X_0, because Theorem 5.3 of [19] implies that $K_0(C^*(\mathbb{Z}, X_0, h_0)) \cong \mathbb{Z}^{n+1}$.

We will make systematic use of the suspension flow of a homeomorphism. See the introduction to [6]; also see II.5.5 and II.5.6 of [2]. We reproduce the definition here.

Definition 1.2. Let $g \colon X_0 \to X_0$ be a homeomorphism of a compact Hausdorff space. Define commuting actions of $\mathbb{R}$ and $\mathbb{Z}$ on $X_0 \times \mathbb{R}$ by

$$t \cdot (x, s) = (x, \, s + t) \quad \text{and} \quad n \cdot (x, s) = (g^n(x), \, s - n)$$

for $x \in X_0$, $s, t \in \mathbb{R}$, and $n \in \mathbb{Z}$. Let $X = (X_0 \times \mathbb{R})/\mathbb{Z}$, and for $x \in X_0$ and $s \in \mathbb{R}$ let $[x, s]$ denote the image of (x, s) in X. The action of $\mathbb{R}$ on $X_0 \times \mathbb{R}$ descends to an action of $\mathbb{R}$ on X, given by the homeomorphisms $h_t([x, s]) = [x, \, s + t]$ for $x \in X_0$ and $s, t \in \mathbb{R}$, called the *suspension flow* of g. We refer to h_t as the *time t map* of the suspension flow.

We will need the following properties of extensions of dynamical systems. They are surely well known. However, we know of no reference for Part (2) except for Theorem A.10 of [5] (although the reverse result, going from Y to X, is Corollary IV.1.9 of [2]). Part (1) is in Theorem A.10 of [5] and also in VI.5.21 of [2], but we give the short proof here for completeness. The proof of Part (2) follows the proof of Theorem 2.6 of [24].

Lemma 1.3. Let $g \colon X \to X$ and $h \colon Y \to Y$ be homeomorphisms of compact Hausdorff spaces, and let $p \colon X \to Y$ be a surjective map such that $h \circ p = p \circ g$. (Thus, g is an extension of h.) Let $N \subset Y$ be

$$N = \{y \in Y \colon \operatorname{card}(p^{-1}(y)) > 1\}.$$

Then:

(1) If h is minimal and $X \setminus p^{-1}(N)$ is dense in X, then g is minimal.
(2) If h has a unique ergodic measure ν, and $\nu(N) = 0$, then g is uniquely ergodic.

Proof. For the first part, let $K \subset X$ be closed, invariant, and not equal to X. Then $X \setminus K$ is open and nonempty, so contains a point x of $X \setminus p^{-1}(N)$. By the definition of N, we have $p(x) \notin p(K)$. Since $p(K)$ is a compact invariant subset of Y, we have $p(K) = \varnothing$, whence $K = \varnothing$.

Now we prove the second part. We define a g-invariant Borel probability measure μ on X by $\mu(E) = \nu(p(E \cap [X \setminus p^{-1}(N)])$ for a Borel set $E \subset X$. Let λ be any other g-invariant Borel probability measure on X. Then $F \mapsto \lambda(p^{-1}(F))$ is an h-invariant Borel probability measure on Y, whence $\lambda(p^{-1}(F)) = \nu(F)$ for every Borel set $F \subset Y$. In particular, $\lambda(p^{-1}(N)) = 0$. Considering subsets of $X \setminus p^{-1}(N)$, it now follows easily that $\lambda = \mu$. ∎

The following lemma is contained in Proposition V.2 of [5]. For the convenience of the reader, we give the proof here. Also, note the relevance of Corollary 2.8 of [7], although it won't in fact be used in the proof.

Lemma 1.4. Let g be a Denjoy homeomorphism of S^1 with rotation number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Let $g_0 \colon X_0 \to X_0$ be the restriction of g to its unique minimal set. Let $t \in \mathbb{R}$, and let $h_t \colon X \to X$ be the time t map of the suspension flow of g_0. Then the following are equivalent:

(1) 1, $t\alpha$, and t are linearly independent over $\mathbb{Q}$.
(2) h_t is minimal.
(3) h_t is uniquely ergodic.

Proof. Let $Q = Q(g_0) \subset S^1$ be the countable set of Definition 3.5 of [19]. First, observe that there is a surjection $p_0 \colon S^1 \to X_0$ such that, with R_α being the rotation by α on S^1, we have $R_\alpha \circ p_0 = p_0 \circ g_0$. That is, g_0 is an extension of R_α in the sense in Lemma 1.3. Moreover, the points in S^1 whose inverse images are not unique are exactly the elements of Q. Let $Y = (S^1 \times \mathbb{R})/\mathbb{Z}$ be the space of the suspension flow of R_α, and let $k_t \colon Y \to Y$ be the time t map of this flow. Then h_t is an extension of k_t. Let $p \colon X \to Y$ be the extension map. The set N of points in Y whose inverse images under p are not unique is $\{[y, s] \in Y : y \in Q\}$.

Define $f \colon Y \to (\mathbb{R}/\mathbb{Z})^2$ by $f([y, s]) = (y + s\alpha + \mathbb{Z},\, s + \mathbb{Z})$. Then $f \circ k_t \circ f^{-1}$ is the homeomorphism of $(\mathbb{R}/\mathbb{Z})^2$ given by $(y_1, y_2) \mapsto (y_1 + (t\alpha + \mathbb{Z}),\, y_2 + (t + \mathbb{Z}))$.

Suppose that 1, $t\alpha$, and t are not linearly independent over $\mathbb{Q}$. Then $f \circ k_t \circ f^{-1}$ has two disjoint nonempty closed invariant sets Z_1 and Z_2. (In fact there are uncountably many.) So $(f \circ p)^{-1}(Z_1)$ and $(f \circ p)^{-1}(Z_2)$ are disjoint nonempty closed h_t-invariant subsets of X. Thus h_t is not minimal. Since each of these sets carries an invariant Borel probability measure, h_t is not uniquely ergodic either.

Now suppose that 1, $t\alpha$, and t are linearly independent over $\mathbb{Q}$. Then k_t is minimal and uniquely ergodic, with Lebesgue measure as the unique invariant measure. The set $p_0^{-1}(Q)$ is countable, so that $X_0 \setminus p_0^{-1}(Q)$ is dense in X_0. Therefore $X \setminus p^{-1}(N)$ is dense in X. It follows from Lemma 1.3(1) that minimality of k_t implies minimality of h_t. Moreover, N has measure zero because Q is countable. So it follows from Lemma 1.3(2) that unique ergodicity of k_t implies unique ergodicity of h_t. ∎

Theorem 1.5. Let G_0 be a finitely generated free abelian group, let $\omega\colon \mathbb{Z}\oplus G_0 \to \mathbb{R}$ be a homomorphism such that $\omega(1,0) = 1$ and $\omega(\mathbb{Z}\oplus G_0)$ has rank at least three. Then there exists a restricted Denjoy homeomorphism $h_0\colon X_0 \to X_0$ with cut number $\mathrm{rank}(G_0) - 1$, and a number $t > 0$, such that the time t map $h\colon X \to X$ of the suspension flow of h_0 has the following properties:

- h is minimal and uniquely ergodic.
- X is connected.
- There is an isomorphism $K_0(C^*(\mathbb{Z}, X, h)) \cong \mathbb{Z}\oplus G_0$ which sends $(1, 0)$ to $[1]$, sends $K_0(C^*(\mathbb{Z}, X, h))_+$ to $\{0\} \cup \{g \in \mathbb{Z}\oplus G_0 \colon \omega(g) > 0\}$, and identifies ω with the map $K_0(C^*(\mathbb{Z}, X, h)) \to \mathbb{R}$ induced by the unique tracial state (coming from the unique ergodic measure on X).
- There is an isomorphism $K_1(C^*(\mathbb{Z}, X, h)) \cong \mathbb{Z}\oplus G_0$.

Proof. Set $G = \mathbb{Z}\oplus G_0$. We identify G_0 with $0 \oplus G_0 \subset G$ in the obvious way.

We first claim that there is a direct sum decomposition $G_0 = H_0 \oplus H_1 \oplus H_2$ with the following properties:

- $H_0 \subset \mathrm{Ker}(\omega)$.
- $\omega|_{H_1\oplus H_2}$ is injective.
- $\omega(H_1) \subset \mathbb{Q}$.
- H_1 has rank zero or one.
- $\omega(H_2) \cap \mathbb{Q} = \{0\}$.

To prove this, first observe that $\omega(G_0)$ is finitely generated and torsion free, so that $\omega|_{G_0}$ has a right inverse $f_0\colon \omega(G_0) \to G_0$. Thus there is a direct sum decomposition $G_0 = H_0 \oplus (f_0 \circ \omega)(G_0)$ with $H_0 = G_0 \cap \mathrm{Ker}(\omega)$. If $\omega(G_0) \cap \mathbb{Q} = \{0\}$, then take $H_1 = \{0\}$ and $H_2 = (f_0 \circ \omega)(G_0)$. Otherwise, let $\pi\colon \mathbb{R} \to \mathbb{R}/\mathbb{Q}$ be the quotient map. Then $(\pi\circ\omega)(G_0)$ is again finitely generated and torsion free, so that $\pi|_{\omega(G_0)}$ has a right inverse $f_1\colon (\pi \circ \omega)(G_0) \to \omega(G_0)$. Set $H_1 = f_0(\mathrm{Ker}(\pi|_{\omega(G_0)}))$ and $H_2 = (f_0 \circ f_1 \circ \pi \circ \omega)(G_0)$, giving $(f_0 \circ \omega)(G_0) = H_1 \oplus H_2$. We have $\omega(H_2) \cap \mathbb{Q} = \{0\}$ because $\pi|_{(f_1\circ\pi\circ\omega)(G_0)}$ is injective. Also, $\omega(H_1)$ is a nonzero finitely generated subgroup of $\mathbb{Q}$, and therefore has rank one. This proves the claim.

Set $m = \mathrm{rank}(H_1 \oplus H_2)$ and $n = \mathrm{rank}(H_0)$. Write

$$\omega(G_0) = \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2 + \cdots + \mathbb{Z}\beta_m,$$

with $\beta_1, \beta_2, \ldots, \beta_m$ linearly independent over $\mathbb{Q}$. If $H_1 \neq \{0\}$, then, using the direct sum decomposition $\omega(G_0) = \omega(H_1) \oplus \omega(H_2)$, we may assume $\beta_m \in \mathbb{Q}$. We may also obviously assume $\beta_j > 0$ for $1 \le j \le m$. Since $\mathrm{rank}(\omega(G)) \ge 3$, the numbers $1, \beta_1, \beta_2$ must be linearly independent over $\mathbb{Q}$.

Choose $g_1, \ldots, g_m \in H_1 \oplus H_2$ such that $\omega(g_j) = \beta_j$ for $1 \le j \le m$. Then $g_1, \ldots, g_m$ form a basis for $H_1 \oplus H_2$. Further let $g_{m+1}, \ldots, g_{m+n}$ form a basis for H_0. Choose an integer $N > 1$ so large that $N^{-n}\beta_j/\beta_1 < 1$ for $2 \le j \le m$.

Set

$$\gamma_1 = 0, \quad \gamma_2 = \frac{\beta_3}{N^n\beta_1}, \quad \gamma_3 = \frac{\beta_4}{N^n\beta_1}, \quad \ldots, \quad \gamma_{m-1} = \frac{\beta_m}{N^n\beta_1},$$

$$\gamma_m = \frac{1}{N^n}, \quad \gamma_{m+1} = \frac{1}{N^{n-1}}, \quad \ldots, \quad \gamma_{m+n-1} = \frac{1}{N}.$$

By the choice of N, we have $\gamma_j \in (0, 1)$ for $2 \le j \le m+n-1$. Let $\overline{\gamma}_j$ be the image of γ_j in $S^1 = \mathbb{R}/\mathbb{Z}$. Set $\alpha = \beta_2/(N^n\beta_1)$, and let $\overline{\alpha}$ be its image in S^1. Define $Q \subset S^1$ by

$$Q = \{\overline{\gamma}_j + l\overline{\alpha}\colon 1 \le j \le m+n-1 \text{ and } l \in \mathbb{Z}\}.$$

Then Q is a countable subset of S^1 which is invariant under the rotation R_α by $\overline{\alpha}$.

We now claim that if $j \neq k$ then $\gamma_j - \gamma_k \notin \mathbb{Z} + \alpha\mathbb{Z}$. Set

$$I = \{1, m, m+1, \ldots, m+n-1\}.$$

If $j, k \in I$, then $\gamma_j - \gamma_k$ is rational and $0 < |\gamma_j - \gamma_k| < 1$, so $\gamma_j - \gamma_k \notin \mathbb{Z} + \alpha\mathbb{Z}$. If $j, k \notin I$, and $\gamma_j - \gamma_k \in \mathbb{Z} + \alpha\mathbb{Z}$, multiply by $N^n\beta_1$. We get $\beta_{j+1} - \beta_{k+1} \in N^n\beta_1\mathbb{Z} + \beta_2\mathbb{Z}$. This is a linear dependence of β_1, β_2, β_{j+1}, and β_{k+1} over $\mathbb{Q}$, a contradiction because $j+1, k+1 \geq 3$. Now suppose that $j \in I$ but $k \notin I$, and that $j \neq 1$. Write $j = m + l$ with $0 \leq l \leq n-1$. If $\gamma_j - \gamma_k \in \mathbb{Z} + \alpha\mathbb{Z}$, multiply by $N^n\beta_1$, getting $N^l\beta_1 - \beta_{k+1} \in N^n\beta_1\mathbb{Z} + \beta_2\mathbb{Z}$. Since $k+1 \geq 3$, the numbers β_1, β_2, and β_{k+1} are linearly independent over $\mathbb{Q}$, so this is a contradiction. If instead $j = 1$, the same procedure would give $-\beta_{k+1} \in N^n\beta_1\mathbb{Z} + \beta_2\mathbb{Z}$, a contradiction for the same reason. This completes the proof the claim.

By Remark 2 in Section 3 of [19], there exists a Denjoy homeomorphism $h_0 \colon S^1 \to S^1$ such that $Q(h_0)$, as in Definition 3.5 of [19], is equal to Q. Let X_0 be its unique minimal set. Let $A_0 = C^*(\mathbb{Z}, X_0, h_0)$ (called D_{h_0} in [19]). By Proposition 4.2 of [19], the algebra A_0 has a unique tracial state τ_0. By Theorem 5.3 and Lemma 6.1 of [19], there is an isomorphism $\rho_0 \colon \mathbb{Z}^{m+n} \to K_0(A_0)$ for which, in terms of the standard generators $\delta_1, \ldots, \delta_{m+n}$ of $\mathbb{Z}^{m+n}$, one has $(\tau_0)_*(\rho_0(\delta_1)) = \alpha$, $(\tau_0)_*(\rho_0(\delta_j)) = \gamma_j$ for $2 \leq j \leq m+n-1$, and $\rho_0(\delta_{m+n}) = [1]$.

We define a different isomorphism $\rho_1 \colon \mathbb{Z}^{m+n} \to K_0(A_0)$ as follows. We set $\rho_1(\delta_1) = \rho_0(\delta_m)$ and $\rho_1(\delta_j) = \rho_0(\delta_{j-1})$ for $2 \leq j \leq m$, and we further set

$$\rho_1(\delta_{m+1}) = \rho_0(\delta_{m+1}) - N\rho_0(\delta_m), \quad \rho_1(\delta_{m+2}) = \rho_0(\delta_{m+2}) - N^2\rho_0(\delta_m), \quad \ldots,$$
$$\rho_1(\delta_{m+n}) = \rho_0(\delta_{m+n}) - N^n\rho_0(\delta_m).$$

This gives:

- $(\tau_0)_*(\rho_1(\delta_1)) = 1/N^n$.
- $(\tau_0)_*(\rho_1(\delta_j)) = \beta_j/(N^n\beta_1)$ for $2 \leq j \leq m$.
- $(\tau_0)_*(\rho_1(\delta_j)) = 0$ for $m+1 \leq j \leq m+n$.

Now take $t = N^n\beta_1$. Since $1, \beta_1, \beta_2$ are linearly independent over $\mathbb{Q}$, it is easy to check that $1, t\alpha, t$ are linearly independent over $\mathbb{Q}$. So the time t map $h \colon X \to X$ of the suspension flow of h_0 is minimal and uniquely ergodic by Lemma 1.4. Also, X is connected by Lemma 1.3 of [7]. Let μ be the unique h-invariant Borel probability measure on X. (It is necessarily obtained following Definition 1.8 of [6] from the unique h_0-invariant Borel probability measure μ_0 on X_0.) Let τ be the corresponding tracial state on $A = C^*(\mathbb{Z}, X, h)$. By Theorem 1.12 of [6], there is an isomorphism $\varphi \colon \mathbb{Z} \oplus K_0(A_0) \to K_0(A)$ such that $\varphi(1, 0) = [1]$ and $\tau_*(\varphi(0, \eta)) = t \cdot (\tau_0)_*(\eta)$ for $\eta \in K_0(A_0)$. We now define $\rho \colon G \to K_0(A)$ on basis elements by $\rho(1, 0) = \varphi(1, 0)$ and $\rho(g_j) = \varphi(0, \rho_1(\delta_j))$ for $1 \leq j \leq m+n$. This defines an isomorphism such that $\rho(1, 0) = [1]$ and $\tau_* \circ \rho = \omega$. It follows from Theorem 4.5(1) of [15] that $\eta \in K_0(A)$ is positive if and only if either $\eta = 0$ or $\tau_*(\eta) > 0$, so ρ is an order isomorphism.

Finally, Theorem 1.12 of [6] also implies $K_1(A) \cong \mathbb{Z} \oplus K_0(A_0) \cong G$ as abelian groups. ∎

Theorem 1.6. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix. Let A_θ be the corresponding (higher dimensional) noncommutative torus, and let τ be its unique tracial state. Suppose that $\operatorname{rank}(\tau_*(K_0(A_\theta))) \geq 3$. Then A_θ is isomorphic to the crossed product by a minimal homeomorphism of a compact connected metric space, obtained as the irrational time map of the suspension flow of a restricted Denjoy homeomorphism.

Proof. We claim that that for every skew symmetric real $d \times d$ matrix (nondegenerate or not), $\mathbb{Z}[1]$ is a direct summand in $K_0(A_\theta)$. We prove this by induction on d. The claim is trivially true for $d = 1$. Suppose it is known for d, and let θ be a skew symmetric real $(d+1) \times (d+1)$ matrix. Let θ_0 be the $d \times d$ upper left corner. Then there is an automorphism α of A_{θ_0}, determined by the requirement that α multiply each of the standard unitary generators of A_{θ_0} by a suitable scalar, such that $A_\theta \cong C^*(\mathbb{Z}, A_{\theta_0}, \alpha)$. (See Notation 1.1 of [17] for the explicit formulas. Also see the unpublished preprint [16].) In particular, α is homotopic to the identity. The Pimsner-Voiculescu exact sequence [18] therefore splits into two short exact sequences. With $\iota\colon A_{\theta_0} \to A_\theta$ being the inclusion map, one of these is

$$0 \longrightarrow K_0(A_{\theta_0}) \xrightarrow{\iota_*} K_0(A_\theta) \longrightarrow K_1(A_{\theta_0}) \longrightarrow 0.$$

The sequence splits because $K_1(A_{\theta_0})$ is free. Thus, $K_0(A_{\theta_0})$ is a summand in $K_0(A_\theta)$, and the map carries the summand $\mathbb{Z}[1_{A_{\theta_0}}]$ in $K_0(A_{\theta_0})$ to $\mathbb{Z}[1_{A_\theta}]$. This proves the claim.

Now let θ be a nondegenerate skew symmetric real $d \times d$ matrix. Use the claim to write $K_0(A_\theta) = \mathbb{Z}[1] \oplus G_0$ for some subgroup $G_0 \subset K_0(A_\theta)$, necessarily isomorphic to $\mathbb{Z}^{2^{d-1}-1}$. Apply Theorem 1.5 with τ_* in place of ω, obtaining $h\colon X_0 \to X_0$ and $h\colon X \to X$ as there. Then h is a minimal homeomorphism, X is a one dimensional compact connected metric space, and $C^*(\mathbb{Z}, X, h)$ has the same Elliott invariant as A_θ. It follows from Theorem 3.5 of [17] (also see the unpublished preprint [16]) that A_θ has tracial rank zero in the sense of [11] (is tracially AF in the sense of [10]; also see [12]), and it follows from Theorem 4.6 of [13] that $C^*(\mathbb{Z}, X, h)$ has tracial rank zero. It is well known that both algebras are simple, separable, nuclear, and satisfy the Universal Coefficient Theorem. Therefore Theorem 5.2 of [12] implies that $A_\theta \cong C^*(\mathbb{Z}, X, h)$. ∎

We point out that one can use the same methods to match the Elliott invariants of other C*-algebras. For example, let $\theta, \gamma \in \mathbb{R}$ be numbers such that $1, \theta, \gamma$ are linearly independent over $\mathbb{Q}$, and let $f\colon S^1 \to \mathbb{R}$ be a continuous function. Let $\alpha_{\theta,\gamma,1,f}$ be the corresponding noncommutative Furstenberg transformation of the irrational rotation algebra A_θ as in Definition 1.1 of [14]. The computation of the Elliott invariant follows from Lemma 1.7 and Corollary 3.5 of [14], and the proof of Theorem 1.6 can be applied to find a restricted Denjoy homeomorphism and a minimal irrational time map $h\colon X \to X$ of its suspension flow such that $C^*(\mathbb{Z}, X, h)$ has the same Elliott invariant as $C^*(\mathbb{Z}, A_\theta, \alpha_{\theta,\gamma,1,f})$.

2. The rank of the range of the trace

In this section, we determine when $\operatorname{rank}(\tau_*(K_0(A_\theta))) = 2$. This is possible for a simple higher dimensional noncommutative torus A_θ.

Example 2.1. Let $\theta_0 \in \mathbb{R} \setminus \mathbb{Q}$ satisfy a nontrivial quadratic equation with integer coefficients, and let $\theta_1, \ldots, \theta_n \in (\mathbb{Q} + \mathbb{Q}\theta_0) \setminus \mathbb{Q}$. Then the tensor product $A = A_{\theta_1} \otimes \cdots \otimes A_{\theta_n}$ of irrational rotation algebras is a simple higher dimensional noncommutative torus such that $\tau_*(K_0(A)) \subset \mathbb{Q} + \mathbb{Q}\theta_0$.

It seems not to be possible to obtain A as a crossed product in the manner of Theorem 1.6.

We give Elliott's description of $\tau_*(K_0(A_\theta))$. First, we need some notation. We regard the skew symmetric real $d \times d$ matrix θ as a linear map from $\mathbb{Z}^d \wedge \mathbb{Z}^d$ to $\mathbb{R}$. Following [3], if $\varphi\colon \Lambda^k \mathbb{Z}^d \to \mathbb{R}$ and $\psi\colon \Lambda^l \mathbb{Z}^d \to \mathbb{R}$ are linear, we take, by a slight abuse of notation, $\varphi \wedge \psi\colon \Lambda^{k+l} \mathbb{Z}^d \to \mathbb{R}$ to be the functional obtained from the alternating functional on $(\mathbb{Z}^d)^{k+l}$ defined as the antisymmetrization of

$$(x_1, x_2, \dots, x_{k+l}) \mapsto \varphi(x_1 \wedge x_2 \wedge \cdots \wedge x_k)\psi(x_{k+1} \wedge x_{k+2} \wedge \cdots \wedge x_{k+l}).$$

In a similar way, we take $\varphi \oplus \psi\colon \Lambda^k \mathbb{Z}^d \oplus \Lambda^l \mathbb{Z}^d \to \mathbb{R}$ to be $(\xi, \eta) \mapsto \varphi(\xi) + \psi(\eta)$.

Theorem 2.2 (Elliott)**.** Let θ be a skew symmetric real $d \times d$ matrix. Let τ be any tracial state on A_θ. Then $\tau_*(K_0(A_\theta))$ is the range of the "exterior exponential", given in the notation above by

$$\exp_\wedge(\theta) = 1 \oplus \theta \oplus \tfrac{1}{2}\theta \wedge \theta \oplus \tfrac{1}{6}\theta \wedge \theta \wedge \theta \oplus \cdots : \Lambda^{\text{even}} \mathbb{Z}^d \to \mathbb{R}.$$

Proof. See 1.3 and Theorem 3.1 of [3]. ∎

Proposition 2.3. Let θ be a skew symmetric real $d \times d$ matrix. Suppose that A_θ is simple and $\operatorname{rank}(\tau_*(K_0(A_\theta))) = 2$. Then d is even, and there exists $\beta \in \mathbb{R} \setminus \mathbb{Q}$ such that every entry of θ is in $\mathbb{Q} + \mathbb{Q}\beta$. If $d > 2$, then β satisfies a nontrivial quadratic equation with rational coefficients.

Proof. Without loss of generality, $|\theta_{j,k}| < 1$ for all j, k. Since $\operatorname{rank}(\tau_*(K_0(A_\theta))) = 2$, there exists $\beta \in \mathbb{R} \setminus \mathbb{Q}$ such that $\tau_*(K_0(A_\theta)) \subset \mathbb{Q} + \mathbb{Q}\beta$. Let $u_1, \dots, u_d$ be the standard unitary generators of A_θ. For $j \neq k$, the elements u_j and u_k generate a subalgebra isomorphic to $A_{\theta_{j,k}}$, which contains a projection p with $\tau(p) = |\theta_{j,k}|$. Thus $\theta_{j,k} \in \mathbb{Q} + \mathbb{Q}\beta$.

We can now write $\theta = C + \beta D$ for skew symmetric $C, D \in M_d(\mathbb{Q})$.

We claim that simplicity of A_θ implies that D is invertible. First, simplicity implies that θ is nondegenerate, that is, there is no $x \in \mathbb{Q}^d \setminus \{0\}$ such that $\langle x, \theta y\rangle \in \mathbb{Q}$ for all $y \in \mathbb{Q}^d$. This is essentially in [23], and in the form given it appears as Lemma 1.7 and Theorem 1.9 of [17]. (Also see the unpublished preprint [16].) Now suppose D is not invertible. Then there exists $x \in \mathbb{Q}^d \setminus \{0\}$ such that $Dx = 0$. For every $y \in \mathbb{Q}^d$ we then have

$$\langle x, \theta y\rangle = \langle x, Cy\rangle + \beta\langle x, Dy\rangle = \langle x, Cy\rangle - \beta\langle Dx, y\rangle.$$

The first term is in $\mathbb{Q}$ and the second is zero, contradicting nondegeneracy of θ. This proves the claim.

Corollary 1 to Theorem 6.3 of [8] now implies that d is even.

Now let $d > 2$. Then $d \geq 4$. Regard D as a map $\mathbb{Z}^d \wedge \mathbb{Z}^d \to \mathbb{Q}$. We claim that, as a map $\Lambda^4 \mathbb{Z}^d \to \mathbb{Q}$, we have $D \wedge D \neq 0$. It is equivalent to prove this with $\mathbb{Q}^d$ in place of $\mathbb{Z}^d$. Since $\mathbb{Q}$ is a field, Theorem 6.3 of [8] allows us to assume that $D = \operatorname{diag}(S, S, \dots, S)$ with $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. (There are no zero diagonal blocks since D is invertible.) Letting $\delta_1, \delta_2, \dots, \delta_d$ be the standard basis vectors for $\mathbb{Q}^d$, a simple calculation now shows that $(D \wedge D)(\delta_1 \wedge \delta_2 \wedge \delta_3 \wedge \delta_4) = \frac{1}{3}$. This proves the claim.

It remains to show that β satisfies a nontrivial quadratic equation. By Theorem 2.2, the range of $\frac{1}{2}\theta \wedge \theta$ is contained in $\mathbb{Q} + \mathbb{Q}\beta$. Choose $\xi \in \Lambda^4 \mathbb{Z}^d$ such that $(D \wedge D)\xi \neq 0$. Then

$$(\theta \wedge \theta)\xi = (C \wedge C)\xi + \beta(C \wedge D + D \wedge C)\xi + \beta^2 (D \wedge D)\xi.$$

Except for $\beta^2 (D \wedge D)\xi$, all terms on both sides of this equation are known to be in $\mathbb{Q} + \mathbb{Q}\beta$. Since $(D \wedge D)\xi \in \mathbb{Q} \setminus \{0\}$, it follows that $\beta^2 \in \mathbb{Q} + \mathbb{Q}\beta$. This completes the proof. ∎

Corollary 2.4. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix, with d odd. Then A_θ is isomorphic to the crossed product by a minimal homeomorphism of a compact connected metric space, obtained as the irrational time map of the suspension flow of a restricted Denjoy homeomorphism.

Proof. Combine Theorem 1.6 and Proposition 2.3. ∎

Corollary 2.5. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix, with $d \geq 4$ even. Suppose the field generated by the entries of θ does not have degree 2 over $\mathbb{Q}$. Then A_θ is isomorphic to the crossed product by a minimal homeomorphism of a compact connected metric space, obtained as the irrational time map of the suspension flow of a restricted Denjoy homeomorphism.

Proof. Again, combine Theorem 1.6 and Proposition 2.3. ∎

3. The three dimensional case

By Corollary 2.4, every the odd dimensional noncommutative torus is isomorphic to the crossed product by a minimal irrational time map of the suspension flow of a restricted Denjoy homeomorphism. For the three dimensional case, there is also a reverse result.

Lemma 3.1. Let θ be a skew symmetric real 3×3 matrix,

$$\theta = \begin{pmatrix} 0 & \theta_{1,2} & \theta_{1,3} \\ -\theta_{1,2} & 0 & \theta_{2,3} \\ -\theta_{1,3} & -\theta_{2,3} & 0 \end{pmatrix}.$$

Then θ is nondegenerate (in the sense used in the proof of Proposition 2.3) if and only if $\dim_{\mathbb{Q}}(\mathrm{span}_{\mathbb{Q}}(1, \theta_{1,2}, \theta_{1,3}, \theta_{2,3})) \geq 3$.

Proof. If $\dim_{\mathbb{Q}}(\mathrm{span}_{\mathbb{Q}}(1, \theta_{1,2}, \theta_{1,3}, \theta_{2,3})) \leq 2$, then θ is degenerate by Proposition 2.3.

Now suppose that $\dim_{\mathbb{Q}}(\mathrm{span}_{\mathbb{Q}}(1, \theta_{1,2}, \theta_{1,3}, \theta_{2,3})) \geq 3$. Then at least two of $\theta_{1,2}, \theta_{1,3}, \theta_{2,3}$ are rationally independent. Suppose $\theta_{1,2}$ and $\theta_{1,3}$ are rationally independent; the other cases are treated similarly. Let $x \in \mathbb{Q}^3$ satisfy $\langle x, \theta y \rangle \in \mathbb{Q}$ for all $y \in \mathbb{Q}^3$. We use the formula

$$\langle x, \theta y \rangle = \theta_{1,2}(x_1 y_2 - x_2 y_1) + \theta_{1,3}(x_1 y_3 - x_3 y_1) + \theta_{2,3}(x_2 y_3 - x_3 y_2).$$

Taking $y = (1, 0, 0)$, we get $-\theta_{1,2} x_2 - \theta_{1,3} x_3 \in \mathbb{Q}$, whence $x_2 = x_3 = 0$. Taking $y = (0, 1, 0)$, we then get $\theta_{1,2} x_1 \in \mathbb{Q}$, whence $x_1 = 0$. Thus $x = 0$, and we have proved that θ is nondegenerate. ∎

Proposition 3.2. Let $h_0 \colon X_0 \to X_0$ be a restricted Denjoy homeomorphism with cut number 2 (Definition 1.1), let $t \in \mathbb{R}$, and let $h \colon X \to X$ be the time t map of the suspension flow of h_0. Suppose that h is minimal. Then $C^*(\mathbb{Z}, X, h)$ is isomorphic to a simple three dimensional noncommutative torus.

Proof. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be the rotation number of a Denjoy homeomorphism of the circle whose restriction to its minimal set is h_0. Lemma 1.4 implies that 1, t, and $t\alpha$ are linearly independent over $\mathbb{Q}$, and that h is uniquely ergodic. Let $\overline{\alpha}$ be the image of α in $\mathbb{R}/\mathbb{Z}$. After a suitable rotation, we may write the set $Q(h_0) \subset S^1$ of Definition 3.5 of [19] as

$$Q(h_0) = \{l\overline{\alpha}\colon l \in \mathbb{Z}\} \cup \{\overline{\gamma} + l\overline{\alpha}\colon l \in \mathbb{Z}\}$$

for some $\overline{\gamma} \in \mathbb{R}/\mathbb{Z}$, and choose $\gamma \in \mathbb{R}$ whose image in $\mathbb{R}/\mathbb{Z}$ is $\overline{\gamma}$. Let τ be the unique tracial state on $C^*(\mathbb{Z}, X, h)$. Combining Theorem 5.3 and Lemma 6.1 of [19] with Theorem 1.12 of [6], we get $K_1(C^*(\mathbb{Z}, X, h)) \cong \mathbb{Z}^4$, and we can find a basis for $K_0(C^*(\mathbb{Z}, X, h))$ consisting of [1] and of three elements g_1, g_2, and g_3 such that $\tau_*(g_1) = t$, $\tau_*(g_2) = t\alpha$, and $\tau_*(g_3) = t\gamma$. Set

$$\theta = \begin{pmatrix} 0 & t & t\alpha \\ -t & 0 & t\gamma \\ -t\alpha & -t\gamma & 0 \end{pmatrix}.$$

Since 1, t, and $t\alpha$ are linearly independent over $\mathbb{Q}$, Lemma 3.1 implies that θ is nondegenerate. Theorem 2.2, together with the fact that A_θ is a simple AT algebra (Theorem 3.8 of [17]; also see the unpublished preprint [16]) implies that A_θ has the same Elliott invariant as $C^*(\mathbb{Z}, X, h)$. Therefore $C^*(\mathbb{Z}, X, h) \cong A_\theta$ as in the proof of Theorem 1.6. ∎

Generally, however, the C*-algebras of minimal time t maps of suspension flows of restricted Denjoy homeomorphisms are not isomorphic to any noncommutative torus.

Proposition 3.3. Let h_0 be a restricted Denjoy homeomorphism with cut number n, let $t \in \mathbb{R}$, and let h be the time t map of the suspension flow of h_0. If $n + 2$ is not a power of 2, then $C^*(\mathbb{Z}, X, h)$ is not isomorphic to any higher dimensional noncommutative torus.

Proof. We have $K_0(C^*(\mathbb{Z}, X, h)) \cong \mathbb{Z}^{n+2}$ as a group by Theorem 5.3 of [19] and Theorem 1.12 of [6], while $K_0(A_\theta) \cong \mathbb{Z}^{2^{d-1}}$ for any skew symmetric real $d \times d$ matrix θ. ∎

Proposition 3.4. Let $d \geq 4$. Then there exists a restricted Denjoy homeomorphism h_0 with cut number $n = 2^{d-1} - 2$, and $t \in \mathbb{R}$, such that the time t map h of the suspension flow of h_0 is minimal, such that $K_*(C^*(\mathbb{Z}, X, h))$ is isomorphic to the K-theory of a d-dimensional noncommutative torus as a graded abelian group, but such that $C^*(\mathbb{Z}, X, h)$ is not isomorphic to any higher dimensional noncommutative torus.

Proof. Choose $t, \alpha, \gamma_1, \gamma_2, \ldots, \gamma_n \in \mathbb{R}$ with $\gamma_1 = 0$ and such that $t, \alpha, \gamma_2, \gamma_3, \ldots, \gamma_n$ are algebraically independent over $\mathbb{Q}$. Let $\overline{\gamma}_j$ be the image of γ_j in $S^1 = \mathbb{R}/\mathbb{Z}$, and let $\overline{\alpha}$ be the image of α in S^1. Define $Q \subset S^1$ by

$$Q = \{\overline{\gamma}_j + l\overline{\alpha}\colon 1 \leq j \leq n \text{ and } l \in \mathbb{Z}\}.$$

Then Q is a countable subset of S^1 which is invariant under the rotation R_α by $\overline{\alpha}$, and, by algebraic independence, we have $\gamma_j - \gamma_k \notin \mathbb{Z} + \alpha\mathbb{Z}$ for $j \neq k$. By Remark 2 in Section 3 of [19], there exists a Denjoy homeomorphism $h_0\colon S^1 \to S^1$ such that $Q(h_0)$, as in Definition 3.5 of [19], is equal to Q. Also write $h_0\colon X_0 \to X_0$ for the corresponding restricted Denjoy homeomorphism. Let τ be the unique tracial state on $C^*(\mathbb{Z}, X_0, h_0)$. By Theorem 5.3 of [19], $\tau_*(K_0(C^*(\mathbb{Z}, X_0, h_0)))$ contains all the numbers $\alpha, \gamma_2, \gamma_3, \ldots, \gamma_n$.

Let h be the time t map of the suspension flow. Then Theorem 1.12 of [6] implies that the range of any tracial state on $K_0(C^*(\mathbb{Z}, X, h))$ contains all the numbers t, $t\alpha$, $t\gamma_2$, $t\gamma_3$, $\ldots$, $t\gamma_n$. By algebraic independence, this range generates a subfield of $\mathbb{R}$ with transcendence degree at least $n+1$ over $\mathbb{Q}$.

If θ is any skew symmetric real $d \times d$ matrix, then Theorem 2.2 implies that the image $\tau_*(K_0(A_\theta))$ of the K-theory under the trace is contained in the subfield of $\mathbb{Q}$ generated by the entries of θ, which has transcendence degree at most $\frac{1}{2}d(d-1)$ over $\mathbb{Q}$. Since $d \geq 4$, we have $n+1 > \frac{1}{2}d(d-1)$, so $C^*(\mathbb{Z}, X, h) \not\cong A_\theta$.

Isomorphism of the K-groups as abelian groups follows from Theorem 5.3 of [19] and Theorem 1.12 of [6]. ∎

There are surely examples in which an isomorphism can't be ruled out by transcendence degree, but can be ruled out by more careful arithmetic.

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