

ON THE SUB-OPTIMAL FEEDBACK CONTROL LAW SYNTHESIS OF UNDERACTUATED SYSTEMS

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ABSTRACT. In this paper, a complete solution on the suboptimal feedback control law synthesis for underactuated systems, based on the general optimization framework of the Dynamic Programming Theory is introduced. Control method proposed keeps the general structure of a suboptimal control approach, while the functional defining performance index is based on the underactuated system energy. Some numerical simulations illustrate our proposed methodology.

Keywords: Non holonomic systems, Passivity, Dynamic Programming Theory, Euler-Lagrange model.

1. **Introduction.** Since the late 1970s, trends in controlling degraded systems (underactuated), yielding a non-holonomic system, has grown interest in many fields.

An underactuated system can be found in different dynamical systems for instance: Flexible systems, degraded controlled systems, manipulator and spacecraft degraded underwater vehicles, or systems designed as underactuated systems due to restrictions of cost weight, and complexity, as well as, some reliability advantages.

Underactuated system control schemes have different targets as regulation, trajectories tracking, obstacles avoidance, stabilization in an equilibrium point designed as a security zone among others. One approach addressing these issues is the optimal control on which present analysis is carried out.

In this paper, a synthesis of a suboptimal control for underactuated systems is proposed, which is based on the complete system energy analysis, the underactuated passivity properties, and the Lyapunov stabilization theory. The main contribution lies on the integration of dynamic programming theory, by introducing a functional defining performance index based on the complete system energy, which is used in the whole closed loop system control. I.e., it is used only one synthesized control law in the complete system workspace.

2. Problem Statement and Preliminaries. The main control problem for vertical underactuated robots is to swing it up to its upright position (top unstable equilibrium position) and stabilize it about the vertical axis. For the swinging up control, Spong and Block [16] used partial feedback linearization techniques and for the balancing and stabilizing controller, used linearization about the desired equilibrium point by a LQR. Nevertheless, a stability analysis is not provided, and the authors used concepts such as. The author used concepts such as partial feedback linearization, zero dynamics, and relative degree.

By combining Lyapunov theory with passivity properties and energy shaping, a nonlinear control for some underactuated systems is proposed by Fantoni and Lozano in [13], where Lyapunov Theory takes an important role in controller design and system convergence analysis. Underactuated systems have been investigated and controlled under different approaches a such system as the Acrobot [16], the Pendubot [18], the rotating pendulum, the cart and pole system, the inertial wheel pendulum [13, 17], and other underactuated systems.

In the analysis and control synthesis framework of non holonomic systems, many approaches have been proposed [7]. Recently, in the optimal control framework there are some interesting works. For instance, in [9], a nonlinear predictive control to design optimal surfaces for sliding model control of underactuated nonlinear surfaces is proposed. The main drawback is that the optimization is achieved by fixing some numerical values and finding some redundant parameters. Consequently, an extension or a generalization of this method seems to be not straightforward.

In [3], an approach to the motion control and trajectory planning of three interconnected links possessing two unactuated joints, is introduced. It is based on the approximation for an "optimal control" with an approximation of the Fourier basis parameters. A performance index involving states and the control input is minimized by using quadratic programming based on the Taylor approximation and a Newton-like method.

In [11], a procedure to swing up a double pendulum mounted on a cart is presented. The trajectory (called quasi-zero torque trajectory) to be followed for the system is obtained by interpolation using splines trough the optimization of an initial trajectory (minimization of the torques applied to the unactuated joints). This trajectory is tracked using a kind of gain scheduling scheme based on a linear quadratic optimal controller (LQR) along the swing up trajectory. A "judiciously" choice of all gains matrices Q and R must be made.

Finally, in [10] a technique solving an optimal feedback control law of a particular system (underactuated Heisenberg system or a non-holonomic integrator) is proposed. This method states the problem as a typical optimal control formulation (hard constraint problem with a performance index based on the control input) and consequently a two point boundary value problem arise by the initial and final values of the system trajectories. Once the initial states are given, the optimal trajectory is evaluated by simple forward integration of the system.

In this contribution, a control methodology is designed for two degree of freedom vertical underactuated robots. Our control approach involves a suboptimal analysis and consequently, a switching criteria between the swing-up control and the stabilization control is not required since only one control law is necessary to be applied.

2.1. Euler-Lagrange model properties and Control. The dynamic equation for the robot manipulator can be obtained via the Newton laws approach, for n -DOF joint analysis, or as in this case, from the Euler-Lagrange .

These equations are obtained by the system Lagrangian equation:

$$\mathcal{L} = \mathcal{K} - \mathcal{U}, \tag{1}$$

where \mathcal{K} are kinetic energy sum for each coordinate, and \mathcal{U} are the potential energy respectively. System Kinetic energy \mathcal{K} is obtained by:

$$\mathcal{K} = \frac{1}{2} \sum_{i=1}^n m_i v_i^2 = \frac{1}{2} \dot{q}^T D(q) \dot{q}, \tag{2}$$

where $m_i \in \mathbb{R}$ are the i -th mass of the i -th link, $v_i \in \mathbb{R}^n$ are the i -th velocity of the i -th link, $\dot{q} \in \mathbb{R}^n$ - are the speed generalized coordinates, and the $n \times n$ matrix $D(q)$ is the inertial matrix.

The system potential energy \mathcal{U} is obtained by:

$$\mathcal{U} = \sum_{i=1}^n m_i h_i g, \tag{3}$$

where $m_i \in \mathbb{R}$ are the i -th mass of the i -th link, $h_i \in \mathbb{R}^n$ are the i -th height of the i -th link respect to mass center and g is a gravitational constant.

From (1), (2) and (3), Euler-Lagrange equations follows:

$$\frac{d}{dt} \left[\frac{\partial \mathcal{L}(q, \dot{q}(t))}{\partial \dot{q}(t)} \right] - \frac{\partial \mathcal{L}(q, \dot{q}(t))}{\partial q(t)} + F(\dot{q}) = \tau, \tag{4}$$

or the equivalent form:

$$\frac{d}{dt} \left[\frac{\partial \mathcal{L}(q, \dot{q})}{\partial \dot{q}_i} \right] - \frac{\partial \mathcal{L}(q, \dot{q})}{\partial q_i} + F_i(\dot{q}) = \tau_i,$$

where $F_i(\dot{q})$ are the tribology vector forces, for the i - th link, $i = 1, 2, 3, \dots, n$ where τ_i are the control forces, and n indicates the degree of freedom (DOF).

From (4) we obtain the generalized Euler-Lagrange equation for manipulator robots as:

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) + F(\dot{q}) = \tau. \tag{5}$$

The position coordinates $q \in \mathbb{R}^n$ with associated velocities \dot{q} and accelerations \ddot{q} are controlled via the driving forces $\tau \in \mathbb{R}^n$. The generalized moment of inertia $n \times n$ matrix $D(q)$, the coriolis and centripetal forces $n \times 1$ vector $C(q, \dot{q})\dot{q}$, and the gravitational forces $G(q)$ all vary along the trajectories. Note that (5) is a nonlinear differential equation,

$$C(q, \dot{q})\dot{q} = \dot{D}(q)\dot{q} - \frac{1}{2} \frac{\partial}{\partial q} [\dot{q}^T D(q) \dot{q}], \tag{6}$$

$$G(q) = \frac{\partial \mathcal{U}(q)}{\partial q}, \tag{7}$$

where $G(q)$ is a $n \times 1$ gravitational forces vector, and τ is a $n \times 1$ control forces vector. From the Euler-Lagrange formulation one can obtain the mathematical model as:

$$\frac{d}{dt} \begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ D(q)^{-1} [\tau - C(q, \dot{q})\dot{q} - G(q) - F(\dot{q})] \end{bmatrix}. \tag{8}$$

Observe that (8) can be generally written as follows:

$$\dot{x} = f(x) + g(x)u \tag{9}$$

where $x \in M \subseteq \mathbb{R}^{2n}$ and $u \in \mathbb{R}^m$ is the control input. Since our problem formulation is given for stability around an equilibrium point, x_{eq} , then nonlinear system (9) is stated as follows:

$$\dot{\tilde{x}} = f(\tilde{x}) + g(\tilde{x})u \tag{10}$$

where $\tilde{x} = x - x_{eq}$.

Proposition 2.1. *The linear approximation of nonlinear system (10) can be expressed as*

$$\dot{\tilde{x}} = A\tilde{x} + Bu \tag{11}$$

where $A = \frac{\partial f}{\partial \tilde{x}}|_{\tilde{x}=x_{eq}}$, and $B = \frac{\partial g}{\partial u}|_{\tilde{x}=x_{eq}}$. The pair (A, B) is controllable and there exists a matrix $Q = HH^T$ such that the pair (A, H) is observable.

2.2. Properties of Euler-Lagrange systems. The dynamic equation for manipulator robots (4) have the following interesting properties. Inertial matrix $D(q)$ is positive definite, and $D(q)$ and $C(q, \dot{q})$ have the following properties [12, 15]:

- There exists some positive constant α such that

$$D(q) \geq \alpha I \quad \forall q \in R^n,$$

where I denotes the $n \times n$ identity matrix. Then, matrix $D(q)^{-1}$ exists and it is positive definite.

- The matrix $C(q, \dot{q})$ have a relationship with the inertial matrix as follows:

$$z^T \left[\frac{1}{2} \dot{D}(q) - C(q, \dot{q}) \right] z = 0 \quad \forall q, \dot{q}, z \in R^n.$$

In fact, $\frac{1}{2} \dot{D}(q) - C(q, \dot{q})$ is the skew-symmetric property. And $C(q, \dot{q})$ is a matrix, univocally defined by $D(q)$, which satisfies

$$\dot{D}(q) = C(q, \dot{q}) + C(q, \dot{q})^T.$$

Furthermore skew-symmetric matrix property satisfies the relationship:

$$\dot{q}^T \left[\frac{1}{2} \dot{D}(q) - C(q, \dot{q}) \right] \dot{q} = 0 \quad \forall q, \dot{q} \in R^n.$$

- The Euler-Lagrange system have the total energy as:

$$\mathcal{E} = \mathcal{K} + \mathcal{U} \tag{12}$$

which is a Lagrangian like equation \mathcal{L} . By differentiating (12) we obtain

$$\begin{aligned} \dot{\mathcal{E}} &= \dot{q}^T D(q, \dot{q}) \dot{q} + \frac{1}{2} \dot{q}^T \dot{D}(q, \dot{q}) \dot{q} + \dot{q}^T G(q) \\ &= \dot{q}^T (-C(q, \dot{q}) - G(q) + \tau) + \frac{1}{2} \dot{q}^T \dot{D}(q, \dot{q}) \dot{q} + \dot{q}^T G(q) \\ &= \dot{q}^T \tau. \end{aligned} \tag{13}$$

- From the passivity property

$$V(x) - V(x_0) \leq \int_0^t y^T(\tilde{x})u(\tilde{x})d\tilde{x}, \tag{14}$$

where $V(x)$ is a storage function, $y(\tilde{x})$ is the output, and $u(\tilde{x})$ is the input of the system. By using (13) in the Euler-Lagrange system, energy function \mathcal{E} as the storage function, the following holds:

$$\mathcal{E}(t) - \mathcal{E}(0) \leq \int_0^t \dot{q}^T \tau dt \tag{15}$$

where \dot{q} is the output, and τ is the input of the system, i.e. system verifies the passivity property.

Then, these Euler-Lagrange system properties will be useful in establishing the stability control analysis.

3. The Hamilton-Jacobi Equation. Some methodologies have been proposed in different underactuated systems control schemes, for instance, nonlinear feedback control, open loop control, small amplitude periodically time varying forcing control, continuous time varying feedback law, or optimal approach [1, 4] among others.

In this paper the Dynamic Programming approach [2] is used and the aim is to find an admissible control $u^*(t)$, in order to achieve (10) follows a trajectory x^* by minimizing a performance index:

$$J = \int_0^\infty [f_0(\tilde{x}, u)]dt, \tag{16}$$

where $f_0(\cdot)$ is a positive definite specified function. The control u^* is called the optimal control. Different kinds of approaches have been employed in order to find u^* , see for example [4] and [1]. Specifically we use the sufficient conditions of optimality [1], [8] to derive a suboptimal control law u .

Theorem 3.1. [8] *If there exist a positive definite function $V^*(\tilde{x}(t))$, which is continuously differentiable and satisfies*

$$\left. \frac{dV^*(\tilde{x}(t))}{dt} \right|_{(10)} + f_0(\tilde{x}^*(t), u^*(t)) = 0, \tag{17}$$

then u^ is the optimal control.*

Equation (17) is the well known Hamilton Jacobi Bellman (HJB) equation which immediately leads to an optimal control in feedback form. The following classical result states sufficient conditions for a local minimum for a scalar function.

Theorem 3.2. [4] *Let $L(x, u)$ be a scalar single valued function of the variables x and u . Let $\frac{\partial^2 L(x, u^*)}{\partial u^2}$ exists and be bounded and continuous. Also assume that*

$$\frac{\partial L(x, u^*)}{\partial u} = 0 \tag{18}$$

and

$$\frac{\partial^2 L(x, u^*)}{\partial u^2} > 0. \tag{19}$$

Then u^ is a local minimum.*

4. Control Approach Proposed. Our control method proposes the suboptimal control of underactuated systems based on the system energy balance. As in suboptimal control problems, a functional, which can be minimized according to the desired criteria, is proposed. This functional consists of the total energy equation of the system; (terminal error function) and the energy delivered to the actuator. Despite being formulated in the general optimization framework of the Hamilton-Jacobi theory, the analytical solution of suboptimal control problem is obtained without a computation of any differential equation.

Observe that one problem in dynamic programming approach for nonlinear control systems is to construct or to find the Bellman functional valid. In this contribution, we propose a first approximation of Bellman functional as a Lyapunov functional.

We pointed out that this contribution is relatively easy generalizable to any non holonomic mechanical system.

In order to have a linear regulator control for the linearized system (11), and given the Proposition 2.1 [5], it is possible to solve the algebraic Riccati equation

$$A^T P - PBR^{-1}B^T P + PA = -Q, \tag{20}$$

and to obtain the P matrix, where Q, R are positive definite real symmetric matrices.

4.1. Suboptimal control problem approach. In order to formulate the suboptimal control problem, the following function is defined as :

$$V(\tilde{x}) = \frac{1}{2}k_E \tilde{\mathcal{E}}(\bar{x}_1, \bar{x}_2)^2 + \frac{1}{2}\tilde{x}^T \underbrace{\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}}_A \tilde{x}, \tag{21}$$

$$\tilde{x} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix},$$

where \bar{x}_1 is the error function, $\bar{x}_1 = \tilde{q} = q - q_d$ and $\bar{x}_2 = \dot{\tilde{q}} = \dot{q} - \dot{q}_d$, $\tilde{x} = [\tilde{q}^T \ \dot{\tilde{q}}^T]^T = [\bar{x}_1^T \ \bar{x}_2^T]^T$, the $2n \times 2n$ matrix A is a strictly symmetric and positive definite matrix, where the elements are $n \times n$ positive definite matrices, i.e. $A_{11} = A_{11}^T > 0$, $A_{12} = A_{21}^T > 0$ and $A_{22} = A_{22}^T > 0$, $k_E \in \mathbb{R}$, is a strictly positive definite constant, $\tilde{\mathcal{E}}(q, \dot{q})$ is an energy error function, given by:

$$\tilde{\mathcal{E}}(\bar{x}_1, \bar{x}_2) = \mathcal{E}(\bar{x}_1, \bar{x}_2) - \mathcal{E}_d(\bar{x}_1, \bar{x}_2). \tag{22}$$

By differentiating (21) along the trajectories of (10), it follows:

$$\begin{aligned} \dot{V}(\tilde{x}) &= k_E \tilde{\mathcal{E}}(\bar{x}_1, \bar{x}_2) \dot{\tilde{\mathcal{E}}}(\bar{x}_1, \bar{x}_2) + \frac{1}{2}\tilde{x}^T \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \dot{\tilde{x}} \\ &+ \frac{1}{2}\dot{\tilde{x}}^T \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tilde{x}, \end{aligned} \tag{23}$$

which can be rewritten as:

$$\dot{V}(\tilde{x}) = k_E \tilde{\mathcal{E}}(\bar{x}_1, \bar{x}_2) \dot{\tilde{\mathcal{E}}}(\bar{x}_1, \bar{x}_2) + \tilde{x}^T \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \dot{\tilde{x}}. \tag{24}$$

Remark 4.1. *Given the error dynamics, and by symmetric and definite positive matrix properties, the following holds:*

$$\dot{\tilde{\mathcal{E}}}(q, \dot{q}) = \frac{1}{2}\dot{\tilde{q}}^T D(q)\ddot{\tilde{q}} + \frac{1}{2}\dot{\tilde{q}}^T \dot{D}(q)\dot{\tilde{q}} + \frac{1}{2}\ddot{\tilde{q}}^T D(q)\dot{\tilde{q}} - \dot{\tilde{q}}^T G(q) \tag{25}$$

Since that $D(q) = D^T(q)$, $D(q) > 0$:

$$\dot{\tilde{\mathcal{E}}}(q, \dot{q}) = \ddot{\tilde{q}}^T D(q)\dot{\tilde{q}} + \frac{1}{2}\dot{\tilde{q}}^T \dot{D}(q)\dot{\tilde{q}} - \dot{\tilde{q}}^T G(q). \tag{26}$$

From the Euler-Lagrange equations (5), whether there does not exists tribology forces, system equation follows:

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau, \tag{27}$$

and then (26) can be written as:

$$\dot{\tilde{\mathcal{E}}}(q, \dot{q}) = \dot{\tilde{q}}^T \{ \tau - C(q, \dot{q})\dot{\tilde{q}} - G(q) \} + \frac{1}{2} \dot{\tilde{q}}^T \dot{D}(q)\dot{\tilde{q}} + \dot{\tilde{q}}^T G(q), \tag{28}$$

which implies

$$\begin{aligned} \dot{\tilde{\mathcal{E}}}(q, \dot{q}) &= \dot{\tilde{q}}^T \tau - \dot{\tilde{q}}^T C(q, \dot{q})\dot{\tilde{q}} + \frac{1}{2} \dot{\tilde{q}}^T \dot{D}(q)\dot{\tilde{q}}, \\ &= \dot{\tilde{q}}^T \tau + \underbrace{\dot{\tilde{q}}^T \left[\frac{1}{2} \dot{D}(q) - C(q, \dot{q}) \right] \dot{\tilde{q}}}_{\text{skew-symmetric-property}} \end{aligned} \tag{29}$$

$$\tilde{\mathcal{E}}(q, \dot{q}) = \dot{\tilde{q}}^T \tau.$$

Since $\tilde{\mathcal{E}}$ is the system energy error function, let us verify the passivity property (14), by integrating (29)

$$\int_0^t \dot{\tilde{\mathcal{E}}}(q, \dot{q}) dt = \int_0^t \dot{\tilde{q}}^T \tau dt, \Rightarrow \tilde{\mathcal{E}}(q, \dot{q}) - \tilde{\mathcal{E}}(0, 0) = \int_0^t \dot{\tilde{q}}^T \tau dt, \tag{30}$$

where $\dot{\tilde{q}}$ is take as the input and τ ia the system output.

Then (24) can be rewritten as:

$$\dot{V}(\tilde{x}) = k_E \tilde{\mathcal{E}}(\bar{x}_1, \bar{x}_2) \bar{x}_2^T \tau + \tilde{x}^T \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \dot{\tilde{x}} \tag{31}$$

I.e.

$$\dot{V}(\tilde{x}) = k_E \tilde{\mathcal{E}}(\bar{x}_1, \bar{x}_2) \bar{x}_2^T \tau + \bar{x}_1^T A_{11} \bar{x}_2 + \bar{x}_2^T A_{21} \bar{x}_2 + \bar{x}_1^T A_{12} \dot{\bar{x}}_2 + \bar{x}_2^T A_{22} \dot{\bar{x}}_2 \tag{32}$$

and by introducing (27) in (32), it follows:

$$\begin{aligned} \dot{V}(\tilde{x}) &= k_E \tilde{\mathcal{E}}(\bar{x}_1, \bar{x}_2)^T \tau + \bar{x}_1^T A_{11} \bar{x}_2 + \bar{x}_2^T A_{21} \bar{x}_2 \\ &+ (\bar{x}_1^T A_{12} + \bar{x}_2^T A_{22}) D^{-1}(\bar{x}_1) (\tau - C(\bar{x}_1, \bar{x}_2) \bar{x}_2 - G(\bar{x}_1)) \end{aligned} \tag{33}$$

By applying the Bellman optimization principle, where the performance index, is defined as:

$$J = \frac{1}{2} \int_0^{t_f} \underbrace{(\tilde{x}^T Q \tilde{x} + u^T R u)}_{f_0(\tilde{x}, u)} dt \tag{34}$$

and by applying the dynamic programming, i.e.

$$\min_u \left\{ \left. \frac{dV(\tilde{x})}{dt} \right|_{10} + f_0(\tilde{x}, u) \right\} \tag{35}$$

where $V(\tilde{x})$ is given by (21), $\left. \frac{dV(\tilde{x})}{dt} \right|_{10}$ is given by (33) and f_0 is given by (34), $u \triangleq \tau$, the $2n \times 2n$ matrix Q , verifies $Q = Q^T$, $Q > 0$, the $n \times n$ matrix R is a symmetric and strictly positive definite matrix. Then, (35) can be written as:

$$\min_u \left\{ \begin{aligned} &k_E \tilde{\mathcal{E}}(\bar{x}_1, \bar{x}_2)^T u + \bar{x}_1^T A_{11} \bar{x}_2 + \bar{x}_2^T A_{21} \bar{x}_2 \\ &+ (\bar{x}_1^T A_{12} + \bar{x}_2^T A_{22}) D^{-1}(\bar{x}_1) (u - C(\bar{x}_1, \bar{x}_2) \bar{x}_2 - G(\bar{x}_1)) \\ &+ \tilde{x}^T Q \tilde{x} + u^T R u \end{aligned} \right\}. \tag{36}$$

By using Theorem 3.2, it follows:

$$\frac{\partial}{\partial u} \left\{ \begin{array}{l} k_E \tilde{\mathcal{E}}(\bar{x}_1, \bar{x}_2)^T u + \bar{x}_1^T A_{11} \bar{x}_2 + \bar{x}_2^T A_{21} \bar{x}_2 \\ + (\bar{x}_1^T A_{12} + \bar{x}_2^T A_{22}) D^{-1}(\bar{x}_1) (u - C(\bar{x}_1, \bar{x}_2) \bar{x}_2 - G(\bar{x}_1)) \\ + \tilde{x}^T Q \tilde{x} + u^T R u \end{array} \right\} = 0 \tag{37}$$

$$k_E \tilde{\mathcal{E}}(\bar{x}_1, \bar{x}_2) \bar{x}_2^T + [\bar{x}_1^T A_{12} + \bar{x}_2^T A_{22}] D^{-1}(\bar{x}_1) + u^T R = 0$$

and since, $x^T y = y^T x$:

$$k_E \tilde{\mathcal{E}}(\bar{x}_1, \bar{x}_2) \bar{x}_2 + D^{-1}(\bar{x}_1) [A_{12}^T \bar{x}_1 + A_{22}^T \bar{x}_2] + Ru = 0. \tag{38}$$

Finally, by solving for u :

$$u = -R^{-1} \left\{ k_E \tilde{\mathcal{E}}(\bar{x}_1, \bar{x}_2) \bar{x}_2 + D^{-1}(\bar{x}_1) [A_{12}^T \bar{x}_1 + A_{22}^T \bar{x}_2] \right\}. \tag{39}$$

Remark 4.2. *The choice of k_E is achieved heuristically. The term multiplying by k_E becomes zero when the state tends to the equilibrium point.*

Note: By differentiating once again equation (39) (i.e. taking $\frac{\partial^2}{\partial^2 u} \left(\left. \frac{dV(\tilde{x})}{dt} \right|_{(10)} + f_0(\tilde{x}, u) \right)$), condition (19) holds.

Now we are able to state the following propositions:

Proposition 4.1. *Consider the nonlinear system (10) and the function $V(\tilde{x})$ given by (21). Then a suboptimal control for the system (10) is given by (39).*

Proposition 4.2. *Consider the nonlinear system (10) and the function $V(\tilde{x})$ given by (21). Then the time derivative of the function $V(\tilde{x})$ along to the trajectories of the closed loop system (10) with the suboptimal control law u is negative if:*

$$\|\gamma(\tilde{x})\| - \lambda_{\min} \tilde{Q}(x) < 0 \tag{40}$$

where $\gamma(\tilde{x}) = -\bar{x}_1^T A_{12} D^{-1} (C(\bar{x}_1, \bar{x}_2) \bar{x}_2 + G(\bar{x}_1)) + \bar{x}_1^T A_{11} \bar{x}_2 - \bar{x}_2^T A_{12} D^{-1} (C(\bar{x}_1, \bar{x}_2) \bar{x}_2 + G(\bar{x}_1)) + \bar{x}_2^T A_{21} \bar{x}_2$ and $\lambda_{\min} \tilde{Q}(x) = \sqrt{2\lambda_{\max} Q_{22}^2 - 3\lambda_{\max} Q_{11} Q_{12}}$.

Proof. In the former section we propose that $V(\tilde{x})$ is semi definite positive function, in order to conclude this fact, we need to analyze the closed loop system. This is given by the following equation:

$$\begin{aligned} \dot{V}(\tilde{x}) &= k_E \tilde{\mathcal{E}}(\bar{x}_1, \bar{x}_2) \bar{x}_2^T \tau + \tilde{x}^T \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tilde{x}, \\ &= k_E \tilde{\mathcal{E}}(\bar{x}_1, \bar{x}_2) \bar{x}_2^T \tau + \bar{x}_1^T A_{11} \bar{x}_2 + \bar{x}_2^T A_{21} \bar{x}_2 + \bar{x}_1^T A_{12} \dot{\bar{x}}_2 + \bar{x}_2^T A_{22} \dot{\bar{x}}_2, \end{aligned} \tag{41}$$

then (41) can be:

$$\dot{V}(\tilde{x}) = -\gamma(\tilde{x}) - \tilde{x}^T \tilde{Q}(x) \tilde{x} \tag{42}$$

where

$$\begin{aligned} \gamma(\tilde{x}) &= -\bar{x}_2^T A_{12} D^{-1}(\bar{x}_1) C(\bar{x}_1, \bar{x}_2) \bar{x}_2 - \bar{x}_1^T A_{12} D^{-1}(\bar{x}_1) G(\bar{x}_1) - \bar{x}_2^T A_{22} D^{-1}(\bar{x}_1) C(\bar{x}_1, \bar{x}_2) \bar{x}_2 \\ &\quad - \bar{x}_2^T A_{22} D^{-1}(\bar{x}_1) G(\bar{x}_1) + \bar{x}_1^T A_{11} \bar{x}_2 + \bar{x}_2^T A_{21} \bar{x}_2, \end{aligned} \tag{43}$$

$$\tilde{Q}(x) = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \tag{44}$$

$$Q_{11} = A_{12} R^{-1} D^{-1}(\bar{x}_1) A_{12}$$

$$Q_{12} = Q_{21} = k_E \tilde{\mathcal{E}}(\bar{x}_1, \bar{x}_2) (2R^{-1} D^{-1}(\bar{x}_1) A_{12})$$

$$\begin{aligned} Q_{22} &= k_E \tilde{\mathcal{E}}(\bar{x}_1, \bar{x}_2) R^{-1} \left(k_E \tilde{\mathcal{E}}(\bar{x}_1, \bar{x}_2) + 2D^{-1}(\bar{x}_1) A_{22} \right) \\ &\quad + A_{22} D^{-1}(\bar{x}_1) R^{-1} D^{-1}(\bar{x}_1) A_{22} \end{aligned}$$

and we have that, some positive scalars $\beta_i(i = 0, 1, \dots, 5)$ such that [15]:

$$\begin{aligned} \|D(\bar{x}_1)\| &\geq \lambda_m(D(\bar{x}_1)) > \beta_0 > 0 \\ \|D(\bar{x}_1)\| &\leq \lambda_M(D(\bar{x}_1)) < \beta_1 < \infty \\ \|C(\bar{x}_1, \bar{x}_2)\| &\leq \beta_2 \|\bar{x}_2\| \\ \|G(\bar{x}_1)\| &\leq \beta_3 \\ \|\mathcal{E}(\bar{x}_1, \bar{x}_2)\| &\leq \|\mathcal{K}(\bar{x}_1, \bar{x}_2)\| + \|\mathcal{U}(\bar{x}_1)\| \leq \beta_1 + \beta_4 \leq \beta_5 \\ \|D^{-1}(\bar{x}_1)\| &\geq \lambda_M(D^{-1}(\bar{x}_1)) > \beta_6 > 0 \end{aligned} \tag{45}$$

From $\gamma(\tilde{x})$ we obtain:

$$\begin{aligned} \gamma(\tilde{x}) &= -\tilde{q}^T A_{12} D^{-1}(\bar{x}_1) (C(\bar{x}_1, \bar{x}_2) \dot{\tilde{q}} + G(q)) + \tilde{q}^T A_{11} \dot{\tilde{q}} \\ &\quad - \dot{\tilde{q}}^T A_{12} D^{-1}(\bar{x}_1) (C(\bar{x}_1, \bar{x}_2) \dot{\tilde{q}} + G(q)) + \dot{\tilde{q}}^T A_{21} \dot{\tilde{q}} \end{aligned}$$

then

$$\begin{aligned} \|\gamma(x)\| &\leq \|\bar{x}_1 D^{-1}(\bar{x}_1) A_{12}\| \|C(\bar{x}_1, \bar{x}_2) \bar{x}_2 + G(\bar{x}_1)\| + \|\bar{x}_1\| \|A_{11}\| \|\bar{x}_2\| \\ &\quad + \|\bar{x}_2 D^{-1}(\bar{x}_1) A_{22}\| \|C(\bar{x}_1, \bar{x}_2) \bar{x}_1 + G(\bar{x}_1)\| + \|\bar{x}_1\| \|A_{21}\| \|\bar{x}_2\| \\ &\leq \|\tilde{q}\| \beta_0 \lambda_M A_{12} (\beta_2 \|\bar{x}_2 + \bar{x}_{eq}\| + \beta_3) + \|\bar{x}_1\| \lambda_M A_{11} \|\bar{x}_2\| \\ &\quad + \|\bar{x}_2\| \beta_0 \lambda_M A_{22} (\beta_2 \|\bar{x}_2 + \bar{x}_{2eq}\| + \beta_3) + \|\bar{x}_2\| \lambda_M A_{21} \|\bar{x}_2\| \\ &\leq \eta(\bar{x}_2, \bar{x}_1, \beta_i) \end{aligned}$$

where $\eta(\bar{x}_1, \bar{x}_2, \beta_i)$ is a non negative function.

Since that $\lambda \max(\tilde{Q}(x))$, and $\lambda \min(\tilde{Q}(x))$, is given by:

$$\lambda_M \tilde{Q}(x) = \sqrt{\lambda_M Q_{11}^2 - \lambda_M Q_{11} Q_{12} + 3/2 \lambda_M Q_{22}^2}$$

$$\lambda_m \tilde{Q}(x) = \sqrt{2 \lambda_M Q_{22}^2 - 3 \lambda_M Q_{11} Q_{12}}$$

then $\tilde{Q}(x) > 0$ only if $\lambda_m > 0$, then:

$$2 \lambda_M Q_{22}^2 > 3 \lambda_M Q_{11} Q_{12}$$

and obviously this inequality holds. Then the sufficient condition for $\dot{V}(\tilde{x}) < 0$ is:

$$\|\gamma(\tilde{x})\| - \lambda_m \tilde{Q}(x) < 0$$

thus we conclude that the system (4) in closed loop with (39) is stable. □

5. Illustrative examples: Study Case of 2-DOF systems. In a sake of clarity and compactness, in the following, let us constraint the analysis of the system dimension to only two DOF, i.e. $[\bar{x}_1^T \ \bar{x}_2^T]^T = [q^T \ \dot{q}^T]^T = [q_1 \ q_2 \ \dot{q}_1 \ \dot{q}_2]^T$. and a constant regulation set point, i.e. $x_{1eq} = q_{d1} = c_1$, $x_{2eq} = q_{d2} = c_2$, $x_{3eq} = x_{3eq} = \dot{q}_{d3} = \dot{q}_{d4} = 0$, then

$$\begin{aligned} \bar{x}_1 &= \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{bmatrix}, \quad \bar{x}_2 = \begin{bmatrix} \dot{\tilde{q}}_1 \\ \dot{\tilde{q}}_2 \end{bmatrix}. \\ \tilde{q}_1 &= q_1 - c_1, \quad \dot{\tilde{q}}_1 = \dot{q}_1, \quad \ddot{\tilde{q}}_1 = \ddot{q}_1, \\ \tilde{q}_2 &= q_2 - c_2, \quad \dot{\tilde{q}}_2 = \dot{q}_2, \quad \ddot{\tilde{q}}_2 = \ddot{q}_2. \end{aligned} \tag{46}$$

since and $n = 2$, then the $2n \times 2n$ matrix A reads:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ \underbrace{a_{21} \quad a_{22}}_{A_{11}} & \underbrace{a_{23} \quad a_{24}}_{A_{12}} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ \underbrace{a_{41} \quad a_{42}}_{A_{21}} & \underbrace{a_{43} \quad a_{44}}_{A_{22}} \end{bmatrix}, \tag{47}$$

and a possible $n \times n$ matrix R follows:

$$R = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}. \quad (48)$$

Then, the control law (39), applied to a 2-DOF gives:

$$\begin{aligned} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= -\frac{k_E \tilde{\mathcal{E}}(q, \dot{q})}{r_1 r_2} \begin{bmatrix} r_2 \dot{\tilde{q}}_1 \\ r_1 \dot{\tilde{q}}_2 \end{bmatrix} \\ &- \frac{1}{(r_1 r_2) \det(D(q))} \begin{pmatrix} (r_2 d_{22} a_{13} - r_2 d_{21} a_{14}) \tilde{q}_1 + (r_2 d_{22} a_{23} - r_2 d_{21} a_{24}) \tilde{q}_2 \\ (r_1 d_{11} a_{14} - r_1 d_{21} a_{13}) \tilde{q}_1 + (r_1 d_{11} a_{24} - r_1 d_{12} a_{23}) \tilde{q}_2 \end{pmatrix} \\ &- \frac{1}{(r_1 r_2) \det(D(q))} \begin{pmatrix} (r_2 d_{22} a_{33} - r_2 d_{21} a_{34}) \dot{\tilde{q}}_1 + (r_2 d_{22} a_{43} - r_2 d_{21} a_{44}) \dot{\tilde{q}}_2 \\ (r_1 d_{11} a_{34} - r_1 d_{21} a_{33}) \dot{\tilde{q}}_1 + (r_1 d_{11} a_{44} - r_1 d_{12} a_{43}) \dot{\tilde{q}}_2 \end{pmatrix} \end{aligned} \quad (49)$$

or

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = - \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}^{-1} \left\{ k_E \tilde{\mathcal{E}}(q, \dot{q}) \dot{\tilde{q}} + D^{-1}(q) \left[\begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix}^T \tilde{q} + \begin{bmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{bmatrix}^T \dot{\tilde{q}} \right] \right\}, \quad (50)$$

which can be rewritten as:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -\frac{k_E \tilde{\mathcal{E}}(q, \dot{q})}{r_1 r_2} \begin{bmatrix} r_2 \dot{\tilde{q}}_1 \\ r_1 \dot{\tilde{q}}_2 \end{bmatrix} - \begin{pmatrix} k_{11}(q, \dot{q}) & k_{12}(q, \dot{q}) & k_{13}(q, \dot{q}) & k_{14}(q, \dot{q}) \\ k_{21}(q, \dot{q}) & k_{22}(q, \dot{q}) & k_{23}(q, \dot{q}) & k_{24}(q, \dot{q}) \end{pmatrix} \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \dot{\tilde{q}}_1 \\ \dot{\tilde{q}}_2 \end{bmatrix}, \quad (51)$$

and

$$\begin{pmatrix} k_{11}(q, \dot{q}) & k_{12}(q, \dot{q}) & k_{13}(q, \dot{q}) & k_{14}(q, \dot{q}) \\ k_{21}(q, \dot{q}) & k_{22}(q, \dot{q}) & k_{23}(q, \dot{q}) & k_{24}(q, \dot{q}) \end{pmatrix}^T = \begin{pmatrix} \frac{(r_2 d_{22} a_{13} - r_2 d_{21} a_{14})}{(r_1 r_2) \det(D(q))} & \frac{(r_1 d_{11} a_{14} - r_1 d_{21} a_{13})}{(r_1 r_2) \det(D(q))} \\ \frac{(r_2 d_{22} a_{23} - r_2 d_{21} a_{24})}{(r_1 r_2) \det(D(q))} & \frac{(r_1 d_{11} a_{24} - r_1 d_{12} a_{23})}{(r_1 r_2) \det(D(q))} \\ \frac{(r_2 d_{22} a_{33} - r_2 d_{21} a_{34})}{(r_1 r_2) \det(D(q))} & \frac{(r_1 d_{11} a_{34} - r_1 d_{21} a_{33})}{(r_1 r_2) \det(D(q))} \\ \frac{(r_2 d_{22} a_{43} - r_2 d_{21} a_{44})}{(r_1 r_2) \det(D(q))} & \frac{(r_1 d_{11} a_{44} - r_1 d_{12} a_{43})}{(r_1 r_2) \det(D(q))} \end{pmatrix}. \quad (52)$$

I.e., the control law to be applied for each link can be obtained as follows:

For first link, we have:

$$u_1 = -\frac{k_E \tilde{\mathcal{E}}(q, \dot{q})}{r_1 r_2} r_2 \Delta \dot{\tilde{q}}_1 - \begin{pmatrix} k_{11}(q, \dot{q}) & k_{12}(q, \dot{q}) & k_{13}(q, \dot{q}) & k_{14}(q, \dot{q}) \end{pmatrix} \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \dot{\tilde{q}}_1 \\ \dot{\tilde{q}}_2 \end{bmatrix}, \quad (53)$$

and

$$u_2 = -\frac{k_E \tilde{\mathcal{E}}(q, \dot{q})}{r_1 r_2} r_1 \Delta \dot{\tilde{q}}_2 - \begin{pmatrix} k_{21}(q, \dot{q}) & k_{22}(q, \dot{q}) & k_{23}(q, \dot{q}) & k_{24}(q, \dot{q}) \end{pmatrix} \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \dot{\tilde{q}}_1 \\ \dot{\tilde{q}}_2 \end{bmatrix}. \quad (54)$$

Remark 5.1. Underactuated systems solution. *The control law (53), and (54) have a particular structure, if they are linearized (11) around an equilibrium point (stable or unstable), the closed loop system can be controlled via LQR compensator, and this solution*

is given by (20), since the control law follows a reference given by $\tilde{\mathcal{E}}(q, \dot{q})$, \tilde{q} and $\dot{\tilde{q}}$, and around the equilibrium point, the system solution tends to zero, and then, it follows:

$$\begin{aligned} & \lim_{x \rightarrow x_{eq}} \left(-\frac{r_j k_E \tilde{\mathcal{E}}(q, \dot{q})}{r_1 r_2} \dot{\tilde{q}}_i - \left(k_{i1}(q, \dot{q}) \quad k_{i2}(q, \dot{q}) \quad k_{i3}(q, \dot{q}) \quad k_{i4}(q, \dot{q}) \right) \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \dot{\tilde{q}}_1 \\ \dot{\tilde{q}}_2 \end{bmatrix} \right) \\ &= - \lim_{x \rightarrow x_{eq}} \left(\frac{r_j k_E \tilde{\mathcal{E}}(q, \dot{q})}{r_1 r_2} \dot{\tilde{q}}_i \right) - \lim_{x \rightarrow x_{eq}} \left(\left(k_{i1}(q, \dot{q}) \quad k_{i2}(q, \dot{q}) \quad k_{i3}(q, \dot{q}) \quad k_{i4}(q, \dot{q}) \right) \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \dot{\tilde{q}}_1 \\ \dot{\tilde{q}}_2 \end{bmatrix} \right) \\ &\approx 0 - \lim_{x \rightarrow x_{eq}} \left(\left(k_{i1}(q, \dot{q}) \quad k_{i2}(q, \dot{q}) \quad k_{i3}(q, \dot{q}) \quad k_{i4}(q, \dot{q}) \right) \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \dot{\tilde{q}}_1 \\ \dot{\tilde{q}}_2 \end{bmatrix} \right) \approx -R^{-1} B^T P \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \\ \dot{\tilde{q}}_1 \\ \dot{\tilde{q}}_2 \end{bmatrix}. \end{aligned} \tag{55}$$

Then, the following terms are almost equivalents:

$$\left(k_{i1}(q, \dot{q}) \quad k_{i2}(q, \dot{q}) \quad k_{i3}(q, \dot{q}) \quad k_{i4}(q, \dot{q}) \right) \Big|_{q \rightarrow q_{eq}} \approx R^{-1} B^T P \tag{56}$$

and the control law can be obtained from (56). This is a solution from Riccati equation as for the linear systems, is the LQR solution.¹

6. Numerical Example. In order to illustrate our control law approaches, let us give a numerical simulation which is applied in the 2-DOF robot platforms called the Pendubot and the Rotatory Pendulum.

6.1. The Pendubot system. The pendubot as shown in Figure 1, is a benchmark system [14] for an underactuated robot, consisting of a double pendulum with only an actuator at the first joint. The pendubot has some parameters, the total mass of link 1 is $m_1 = 1.9008m$, the total mass of link 2 is $m_2 = 0.7175m$, the moment of inertia of link 1 is $I_1 = 0.004Kg \cdot m^2$, the moment of inertia of link 2 is $I_2 = 0.005Kg \cdot m^2$, the distance to center of mass of link 1 is $l_{c1} = 0.185m$, the distance to center of mass of link 2 is $l_{c2} = 0.062m$, the length of link 1 is $l_1 = 0.2m$, the length of link 2 is $l_2 = 0.2m$ and the acceleration of gravity constant $g = 9.81m/seg^2$. The model of the motion dynamics is a set of 2 rigid bodies connected and described by a set of generalized coordinates $q \in \mathbb{R}^2$. The derivation of the motion equations is given by (4), and by applying the methods of the Lagrange theory, involving explicit expressions of kinetic energy and potential energy we obtain the standard general equation (5). Where the angular positions are involved in q , and angular velocities in \dot{q} , and the accelerations is \ddot{q} .

¹This solution gives the a_{13} , a_{14} , a_{23} , a_{24} , a_{33} , a_{34} , a_{43} and a_{44} values.

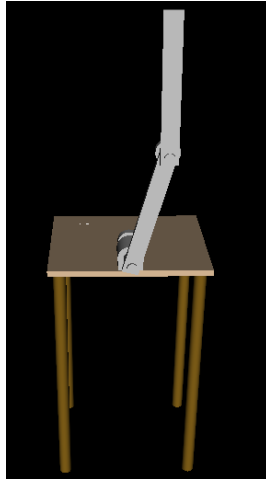


FIGURE 1. Pendubot system.

From the Euler-Lagrange dynamical model one has:

$$\begin{aligned}
 D(q) &= \begin{bmatrix} \theta_1 + \theta_2 + \theta_3 \cos(q_2) & \theta_2 + \theta_3 \cos(q_2) \\ \theta_2 + \theta_3 \cos(q_2) & \theta_2 \end{bmatrix}, \\
 C(q, \dot{q}) &= \theta_3 \sin(q_2) \begin{bmatrix} -\dot{q}_2 & -\dot{q}_1 - \dot{q}_2 \\ -\dot{q}_1 & 0 \end{bmatrix}, \\
 G(q) &= \begin{bmatrix} \theta_4 g \cos(q_1) + \theta_5 g \cos(q_1 + q_2) \\ \theta_5 g \cos(q_1 + q_2) \end{bmatrix}, \\
 q &= \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},
 \end{aligned} \tag{57}$$

where the following five parameter equations are introduced as follows:

$$\begin{cases} \theta_1 = m_1 l_{c1}^2 + m_2 l_1^2 + I_1 \\ \theta_2 = m_2 l_{c2}^2 + I_2 \\ \theta_3 = m_2 l_1 l_{c2} \\ \theta_4 = m_1 l_{c1} + m_2 l_1 \\ \theta_5 = m_2 l_{c2} \end{cases} \tag{58}$$

Whether only u_1 can drive the system (i.e. $u_2 \equiv 0$) such a system is called the Pendubot, while if $u_1 \equiv 0$ the system becomes drive by u_2 and it is called an Acrobot.

Note that $D(q)$ is symmetric. Moreover

$$\begin{aligned}
 d_{11} &= \theta_1 \theta_2 + 2\theta_3 \cos q_2 \\
 &= m_1 l_{c1}^2 + m_2 l_1^2 + I_1 + m_2 l_{c2}^2 + I_2 + 2m_2 l_1 l_{c2} \cos q_2 \\
 &\geq m_1 l_{c1}^2 + m_2 l_1^2 + I_1 + m_2 l_{c2}^2 + I_2 - 2m_2 l_1 l_{c2} \\
 &\geq m_1 l_{c1}^2 + I_1 + I_2 + m_2 (l_1 - l_{c2})^2 > 0,
 \end{aligned} \tag{59}$$

and then

$$\det(D(q)) = \theta_1 \theta_2 - 2\theta_3^2 \cos^2 q_2 = (m_1 l_{c1}^2 + I_1)(m_2 l_{c2}^2 + I_2) + m_2 l_1^2 I_2 + m_2^2 l_1^2 l_{c2}^2 \sin^2 q_2 > 0, \tag{60}$$

Therefore $D(q)$ is positive definite for all q . From (57) it follows that

$$\dot{D}(q) - 2C(q, \dot{q}) = \theta_3 \sin q_2 (2\dot{q}_1 + \dot{q}_2) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{61}$$

which is a skew-symmetric matrix. The potential energy of the pendubot can be defined as

$$\mathcal{U}(q) = \theta_4 \sin q_1 + \theta_5 \sin(q_1 + q_2). \tag{62}$$

Note that \mathcal{U} is related to $g(q)$ as follows:

$$G(q) = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \begin{bmatrix} \theta_4 g \cos q_1 + \theta_5 g \cos(q_1 + q_2) \\ \theta_5 g \cos(q_1 + q_2) \end{bmatrix}. \tag{63}$$

For the planar two-link, when $u \equiv 0$, in the Euler-Lagrange system has four equilibrium points. The first one $(q_1, q_2, \dot{q}_1, \dot{q}_2) = ((\pi/2), 0, 0, 0)$ and the second one $(q_1, q_2, \dot{q}_1, \dot{q}_2) = (-(\pi/2), \pi, 0, 0)$ are two unstable equilibrium points (respectively, top position and mid position). We wish to reach the top position. The third one, $(q_1, q_2, \dot{q}_1, \dot{q}_2) = ((\pi/2), \pi, 0, 0)$ is an unstable equilibrium position, and finally the fourth one $(q_1, q_2, \dot{q}_1, \dot{q}_2) = (-(\pi/2), 0, 0, 0)$ is the stable equilibrium position what e want to avoid them. The total energy $\mathcal{L}(q, \dot{q})$ is different for each of one the four equilibrium positions:

Top positions for both links

$$\mathcal{E}((\pi/2), 0, 0, 0) = \mathcal{E}_{Top} = (\theta_4 + \theta_5)g. \tag{64}$$

Low positions for both links

$$\mathcal{E}(-(\pi/2), 0, 0, 0) = \mathcal{E}_{l_1} = -(\theta_4 + \theta_5)g. \tag{65}$$

Mid position: low for link 1 and up for link 2

$$\mathcal{E}(-(\pi/2), \pi, 0, 0) = \mathcal{E}_{mid} = (-\theta_4 + \theta_5)g. \tag{66}$$

Position: up for link 1 and low for link 2.

$$\mathcal{E}((\pi/2), \pi, 0, 0) = \mathcal{E}_{l_2} = (\theta_4 - \theta_5)g. \tag{67}$$

6.1.1. *Control approach.* Since, the pendubot system, then $u_2 = 0$, and for this case, the desired position is given by the top position² then control approach is given by (53), where the energy error is given by $\tilde{\mathcal{E}} = \mathcal{E}(q, \dot{q}) - \mathcal{E}_{Top}$ $r_1 = r_2$, $d_{21} = \theta_2 + \theta_2 \cos(q_2)$, $d_{22} = \theta_2$, and $\det(D(q)) = \theta_1 \theta_2 - 2\theta_3^2 \cos^2 q_2$, and from (9) the dynamic $g(x)u$ is given by:

$$g(x) = \begin{bmatrix} 0 \\ 0 \\ \frac{\theta_2}{\theta_1 \theta_2 - 2\theta_3^2 \cos(q_2)} \\ \frac{-\theta_2 - \theta_3 \cos(q_2)}{\theta_1 \theta_2 - 2\theta_3^2 \cos(q_2)} \end{bmatrix} \tag{68}$$

applying the proposition (2.1), one obtains

$$B = \begin{bmatrix} 0 \\ 0 \\ \theta_2 \\ \frac{\theta_1 \theta_2 - 2\theta_3^2}{-\theta_2 - \theta_3} \end{bmatrix}$$

By taking the linearization system (11), in the top position of the system, it follows:

²Top position for pendubot is $(q_1, q_2, \dot{q}_1, \dot{q}_2) = (\pi/2, 0, 0, 0)$.

$$\begin{aligned}
 & - \left(\begin{matrix} k_{11}(q, \dot{q}) & k_{12}(q, \dot{q}) & k_{13}(q, \dot{q}) & k_{14}(q, \dot{q}) \end{matrix} \right) \tilde{x} \approx \\
 & - \frac{1}{r_1} \left[\begin{matrix} 0 & 0 & \frac{\theta_2}{\theta_1\theta_2-2\theta_3^2} & \frac{-\theta_2-\theta_3}{\theta_1\theta_2-2\theta_3^2} \end{matrix} \right] \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{12} & p_{22} & p_{23} & p_{24} \\ p_{13} & p_{23} & p_{33} & p_{34} \\ p_{14} & p_{24} & p_{34} & p_{44} \end{bmatrix} \tilde{x} \quad (69)
 \end{aligned}$$

then

$$\begin{aligned}
 & - \frac{1}{r_1} \left[\begin{matrix} 0 & 0 & \frac{\theta_2}{\theta_1\theta_2-2\theta_3^2} & \frac{-\theta_2-\theta_3}{\theta_1\theta_2-2\theta_3^2} \end{matrix} \right] \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{12} & p_{22} & p_{23} & p_{24} \\ p_{13} & p_{23} & p_{33} & p_{34} \\ p_{14} & p_{24} & p_{34} & p_{44} \end{bmatrix} \tilde{x} = \\
 & - \left(\begin{matrix} k_1 & k_2 & k_3 & k_4 \end{matrix} \right) \tilde{x} = -K \tilde{x} \quad (70)
 \end{aligned}$$

and the pendubot system in the top position, with (52), can be approximated at:

$$\left(\begin{matrix} k_{11}(q, \dot{q}) & k_{12}(q, \dot{q}) & k_{13}(q, \dot{q}) & k_{14}(q, \dot{q}) \end{matrix} \right) \Big|_{f(q,\dot{q}) \rightarrow f(0,0)} \approx \left(\begin{matrix} k_1 & k_2 & k_3 & k_4 \end{matrix} \right) \quad (71)$$

where $(k_1 \ k_2 \ k_3 \ k_4) = K$ are obtained from Riccati equation solution (in the linear systems, is the LQR solution), and this solution gives the $a_{13}, a_{14}, a_{23}, a_{24}, a_{33}, a_{34}, a_{43}$ and a_{44} values.

Since the solution of matrix the A gives four equations and seven unknown parameters, we propose the following parameters, $a_{13} = 5, a_{24} = 2$ and $a_{34} = 6$, and then, the parameters of matrix A are obtained as follows:

$$\begin{aligned}
 a_{14} &= \frac{(k_1 - a_{13}B_3)}{B_4}, & a_{33} &= \frac{(k_3 - a_{34}B_4)}{B_3}, \\
 a_{23} &= \frac{(k_2 - a_{24}B_4)}{B_3}, & a_{44} &= \frac{(k_4 - a_{34}B_3)}{B_4},
 \end{aligned} \quad (72)$$

where

$$K = \left(\begin{matrix} k_1 & k_2 & k_3 & k_4 \end{matrix} \right) = \left[\begin{matrix} 0 & 0 & G_1(q) & G_2(q) \end{matrix} \right] P \quad (73)$$

and the parameters $a_{11} = 100, a_{12} = 0, a_{21} = 0$ and $a_{22} = 100$ are proposed such that the matrix $A > 0$.

6.1.2. *Numerical simulation.* In the numerical simulation, the linear gains for LQR solution proposed as:

$$Q = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 11.5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}, \quad R = \text{diag}(2.5).$$

Then, the Riccati equation:

$$P = \begin{bmatrix} 844.7295 & 745.8431 & 155.8352 & 88.8254 \\ 745.8431 & 674.6714 & 138.8006 & 79.3100 \\ 155.8352 & 138.8006 & 29.1716 & 16.6128 \\ 88.8254 & 79.3100 & 16.6128 & 9.5215 \end{bmatrix},$$

and by trial error a possible k_E gain is found as:

$$k_E = 49.8$$

We start the numerical simulation with the following initial conditions:

$$\begin{aligned}
 q_1 &= -\pi/2, & \dot{q}_1 &= 0 \\
 q_2 &= 0, & \dot{q}_2 &= 0
 \end{aligned} \quad (74)$$

6.1.3. *Numerical results.* Figure 2, and Figure 3 show the to be stabilized around time $t \simeq 1$ second, i.e. $q_1 = -\pi/2$, $\dot{q}_1 = 0$, $q_2 = 0$, $\dot{q}_2 = 0$, and the control signal and the total energy is shown in the Figure 4.

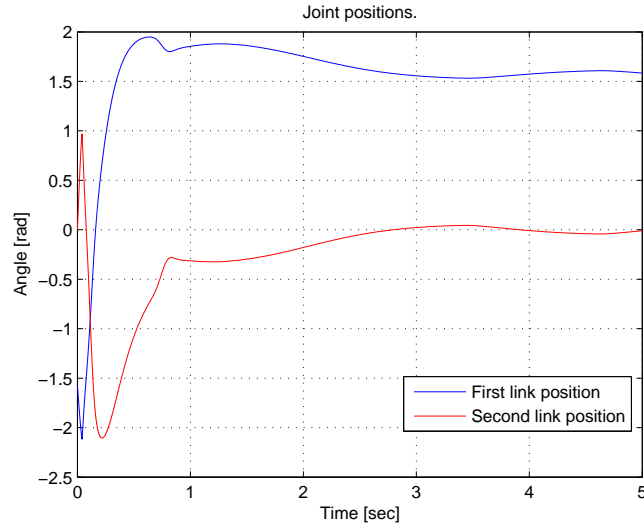


FIGURE 2. The pendubot joint positions.

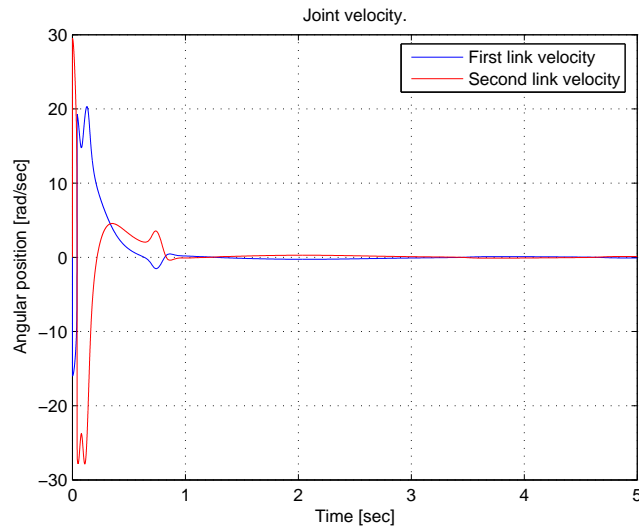


FIGURE 3. The pendubot joint velocities.

6.1.4. *Stability analysis.* By using Proposition 4.1, we conclude the stability for the pendubot system, imposed by $\dot{V}(\tilde{x}) < 0$, the Lyapunov function (32) in closed loop with (53) and $u_2 = 0$, and condition (40), gives the following:

$$\begin{aligned} \lambda_m D^{-1}(q) &= 13.5516, & \lambda_m R^{-1} &= 0.4, & \lambda_M A_{22} &= 16.81, \\ \lambda_M A_{11} &= 100, & \lambda_M A_{12} &= 3.5, \end{aligned}$$

Since are solved, the system parameters are known, we substituting this results in the equation (42), (43) and (44) are solved, and since that β_i are functions strictly positive

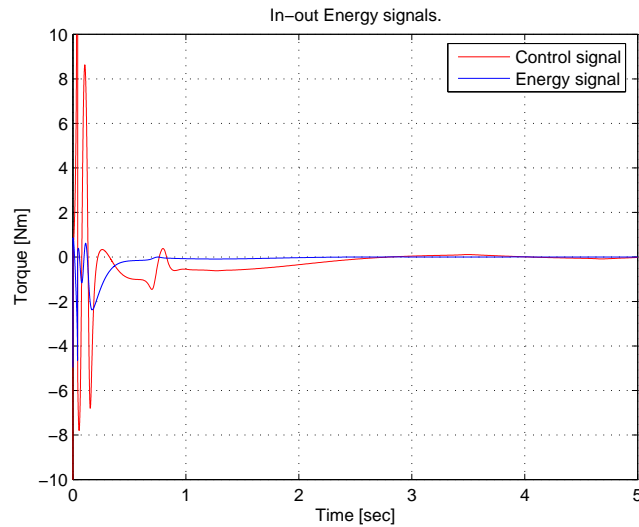


FIGURE 4. The pendubot, in-out energy.

definite, we obtain the following result:

$$\lambda_m \tilde{Q}(x) = 1.4029 \times 10^3 \sqrt{\beta_5^2 + 934.3\beta_5 + 89981.09}$$

and

$$\|\gamma(x)\| = 222.88\beta_2 \|\dot{q}\| \|\tilde{q}\| + 222.88\beta_2 \|\dot{q}\| \|\dot{\tilde{q}}\| - 3.5\beta_3 (\|\tilde{q}\| + \|\dot{\tilde{q}}\|).$$

By applying (40), we start the system at the initial conditions (74), which are sufficient

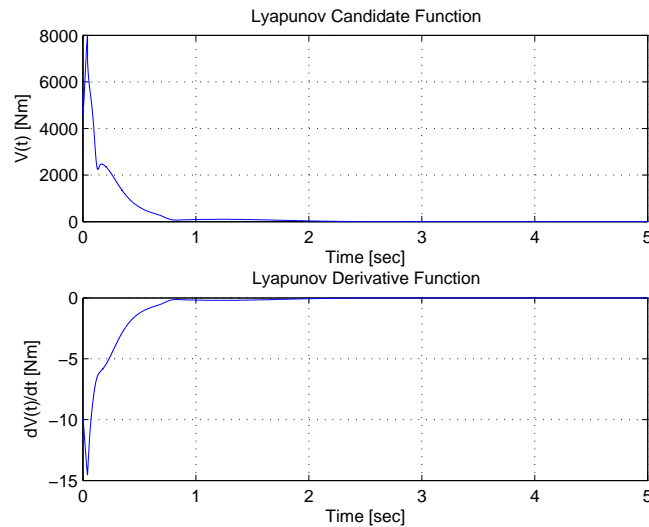


FIGURE 5. Lyapunov functions.

conditions in order to conclude that the closed loop holds on the Lyapunov condition, and this involving an appropriate value for k_E large enough such that \tilde{x} converges into neighborhood $\varepsilon > 0$ with radius $r > 0$ centered in the equilibrium q_d , (the Lyapunov functions can be seen in Figure 5). Another application can be seen in the appendix.

7. Conclusions. In this paper a control approach for underactuated systems is presented. This approach is a general methodology for 2 – *DOF* Euler-Lagrange systems and becomes more involved when the system under analysis has a higher degree (more than 2 – *DOF*). The main reason is because the gains choices is not straightforward. A comparative study shows that the control law does not require a switching control for the equilibrium point. This is because the Hamilton-Jacobi-Bellman principle holds in the whole system workspace, i.e. the Lyapunov function $\dot{V}(\tilde{x}) < 0$ for all $\tilde{x} \in M \subseteq \mathbb{R}^n$. The control design is based on the solution of Hamilton-Jacobi-Bellman equations with a performance index given by (16). This approach is a suboptimal control because it holds only for the sufficient condition to conclude the complete system stability.

The future work aims to apply this approach to underactuated experimental robot platforms.

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Appendix A. The Rotatory Pendulum System. The Rotatory Pendulum as shown in Figure 6, consisting of a double pendulum with an actuator at only the first joint. Using the Rotatory pendulum, one can mainly investigate the set-point regulation, including swinging up and balancing, as well as trajectories tracking. The Rotatory pendulum has been studied as a typical example of underactuated mechanical systems see e.g. [13, 14]. We considered the rotatory pendulum parameters as: the total mass of link 1 $m_1 = 1.9008 m$, the total mass of link 2 $m_2 = 0.7175 m$, the moment of inertia of link 1 $I_1 = 0.004 Kg \cdot m^2$, the moment of inertia of link 2 $I_2 = 0.005 Kg \cdot m^2$, the distance to center of mass of link 1 $l_{c1} = 0.185 m$, the distance to center of mass of link 2 $l_{c2} = 0.062 m$, the length of link 1 $l_1 = 0.2 m$, the length of link 2 $l_2 = 0.2 m$ and the acceleration of gravity $g = 9.81 m/seg^2$.



FIGURE 6. Rotatory pendulum system.

In this case, the kinetic energy $K(q, \dot{q}) = K_1(q, \dot{q}) + K_2(q, \dot{q})$ where $K_1(q, \dot{q})$ y $K_2(q, \dot{q})$ are associated with the rotational arm and the pendulum link respectively as follows:

$$K_1 = \frac{1}{2} I_1 \dot{\theta}_1^2, \quad (75)$$

and the kinetic energy :

$$K_2 = \frac{1}{2} J_2 \dot{\theta}_2^2 + \frac{1}{2} m_2 L_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + \frac{1}{2} m_2 l_2^2 \text{sen}^2 \theta_2 \dot{\theta}_1^2 + m_2 L_1 l_2 \cos \theta_2 \dot{\theta}_1 \dot{\theta}_2. \quad (76)$$

The potential energy of the pendubot can be defined as:

$$U(q) = m_2 l_2 g (\cos \theta_2 - 1) \quad (77)$$

It follows a model stated by (5) where:

$$D(q) = \begin{bmatrix} I_1 + m_2 (L_1^2 + l_2^2 \text{sen}^2 \theta_2) & m_2 l_2 L_1 \cos \theta_2 \\ m_2 l_2 L_1 \cos \theta_2 & J_2 + m_2 l_2^2 \end{bmatrix}, \quad (78)$$

$$C(q, \dot{q}) = \begin{bmatrix} \frac{1}{2} m_2 l_2^2 \text{sen} (2\theta_2) \dot{\theta}_2 & \frac{1}{2} m_2 l_2^2 \text{sen} (2\theta_2) \dot{\theta}_1 - m_2 l_2 L_1 \text{sen} \theta_2 \dot{\theta}_2 \\ -\frac{1}{2} m_2 l_2^2 \text{sen} (2\theta_2) \dot{\theta}_1 & 0 \end{bmatrix}, \quad (79)$$

$$G(q) = \begin{bmatrix} 0 \\ -m_2 l_2 g \text{sen} \theta_2 \end{bmatrix}. \quad (80)$$

Note that $D(q)$ is symmetric. Moreover

$$\begin{aligned} \det(D(q)) &= (I_1 + m_2 (L_1^2 + l_2^2 \sin^2 \theta_2)) (J_1 + m_2 l_1^2) - m_2^2 l_2^2 L_1^2 \cos \theta_2 \\ &= (I_1 + m_2 l_2^2 \sin^2 \theta_2) (J_2 + m_2 l_2^2) + J_2 m_2 L_1^2 + m_2^2 l_2^2 L_1^2 \sin \theta_2 > 0 \end{aligned} \tag{81}$$

Therefore $D(q)$ is positive definite for all θ . From (78) it follows that:

$$\dot{D}(q) - 2C(q, \dot{q}) = m_2 l_2 \left(l_2 \sin(2\theta_2) \dot{\theta}_1 - L_1 \sin \theta_2 \dot{\theta}_2 \right) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{82}$$

which is a skew-symmetric matrix.

For the rotatory pendulum system with $\tau \equiv 0$ in the Euler-Lagrange system note that (4) has two equilibrium points. $(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = (\pi, *, 0, 0)$ the stable equilibrium point and $(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) = (0, *, 0, 0)$ is the stable equilibrium position that also we want to avoid. The control objective is to stabilize the system around its top unstable equilibrium position.

A.1. Control approach. For control design we have the rotatory pendulum system, then $u_2 = 0$, then control approach is given by (53), where $r_1 = r_2$, $d_{21} = I_1 + m_1 (L_1^2 + l_2^2 \sin^2 \theta_2)$, $d_{22} = J_2 + m_2 l_2^2$ and $\det(D(q)) = (I_1 + m_2 l_2^2 \sin^2 \theta_2) (J_2 + m_2 l_2^2) + J_2 m_2 L_1^2 + m_2^2 l_2^2 L_1^2 \sin \theta_2$, and from (9) the dynamic $g(x)u$ is given by:

$$g(x) = \begin{bmatrix} 0 \\ 0 \\ \frac{J_2 + m_2 l_2^2}{(I_1 + m_2 l_2^2 \sin^2 \theta_2)(J_2 + m_2 l_2^2) + J_2 m_2 L_1^2} \\ \frac{-m_2 l_2 L_1 \cos \theta_2}{(I_1 + m_2 l_2^2 \sin^2 \theta_2)(J_2 + m_2 l_2^2) + J_2 m_2 L_1^2} \end{bmatrix} \tag{83}$$

by applying (2.1), we obtain the following equation:

$$B = \begin{bmatrix} 0 \\ 0 \\ \frac{J_2 + m_2 l_2^2}{I_1 (J_2 + m_2 l_2^2) + J_2 m_2 L_1^2} \\ \frac{-m_2 l_2 L_1}{I_1 (J_2 + m_2 l_2^2) + J_2 m_2 L_1^2} \end{bmatrix}$$

A.1.1. Numerical simulation. Using (69), (70), and (71) we propose the follow lineal gains for LQR solution as:

$$Q = \begin{bmatrix} 35.5 & 0 & -17.75 & 0 \\ -17.75 & 35.5 & 0 & 0 \\ 0 & 0 & 35.5 & -17.75 \\ 0 & 0 & -17.75 & 35.5 \end{bmatrix}, \quad R = \text{diag}(2.5),$$

then, we obtain the Riccati equation:

$$P = \begin{bmatrix} 844.7295 & 745.8431 & 155.8352 & 88.8254 \\ 745.8431 & 674.6714 & 138.8006 & 79.3100 \\ 155.8352 & 138.8006 & 29.1716 & 16.6128 \\ 88.8254 & 79.3100 & 16.6128 & 9.5215 \end{bmatrix}$$

and propose the k_E gain as:

$$k_E = 10$$

We start the numerical simulation with the follow initial conditions:

$$\begin{aligned} \theta_1 &= \pi, & \dot{\theta}_1 &= 0 \\ \theta_2 &= 0.5, & \dot{\theta}_2 &= 0 \end{aligned}$$

And we obtain:

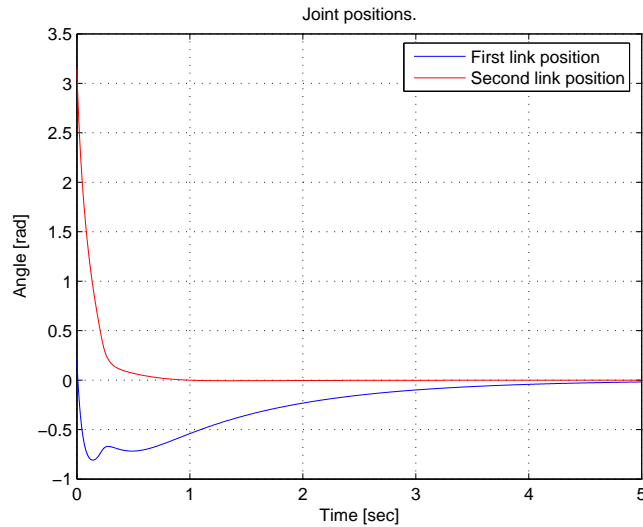


FIGURE 7. The rotatory pendulum, joint positions.

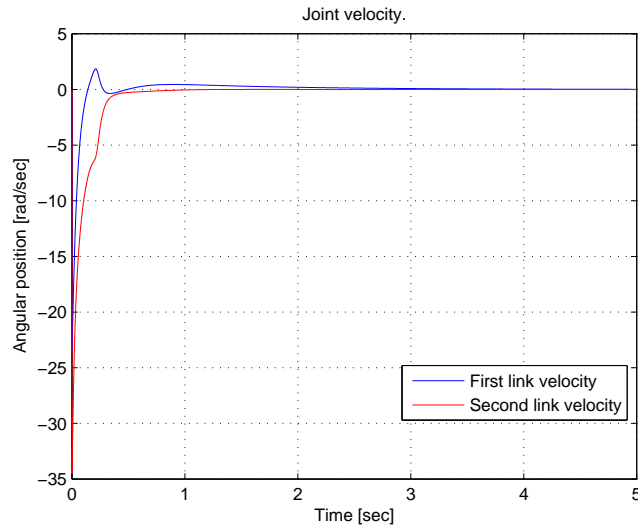


FIGURE 8. The rotatory pendulum joint velocities.

A.1.2. *Stability analysis.* By using the proposition 4.1, we conclude the stability for the Rotatory pendulum system, and we required the Lyapunov function (32) in closed loop with (53) and $u_2 = 0$, and condition (40), and we obtain:

$$\begin{aligned} \lambda_m D^{-1}(q) &= 13.0.258, & \lambda_m R^{-1} &= 0.4, & \lambda_M A_{11} &= 1, \\ \lambda_M A_{12} &= 7.2, & \lambda_M A_{22} &= 6.51, \end{aligned}$$

since we know the system parameters, we substituting this results in the equation (42), (43) and (44), and since that the β_i are constants strictly positive definite, we obtain the follow result:

$$\lambda_m \tilde{Q}(x) = 1.4029 \times 10^3 \sqrt{\beta_5^2 + 934.3\beta_5 + 89981.09}$$

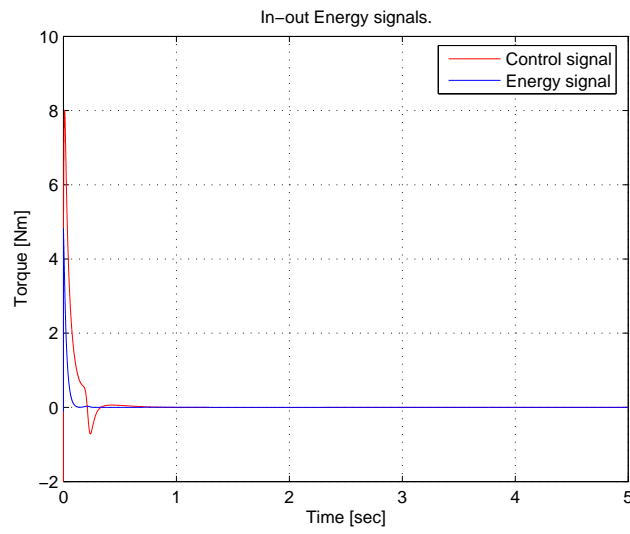


FIGURE 9. The rotatory pendulum, in-out energy.

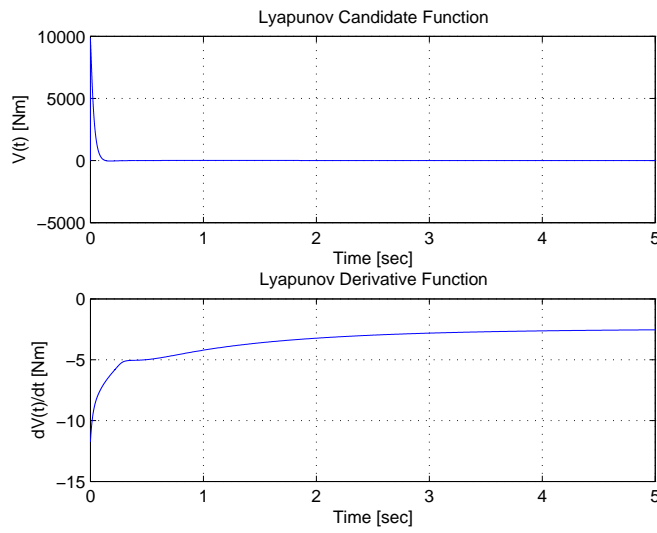


FIGURE 10. Lyapunov function.

and

$$\|\gamma(x)\| = 222.88\beta_2 \|\dot{q}\| \|\tilde{q}\| + 222.88\beta_2 \|\dot{q}\| \|\dot{\tilde{q}}\| - 3.5\beta_3 (\|\tilde{q}\| + \|\dot{\tilde{q}}\|)$$