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Common errors on using convex modelling and linear matrix inequalities for nonlinear control
Errores comunes al usar modelado convexo y desigualdades matriciales lineales en control no lineal

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Abstract

This note discusses common errors on using convex modelling and linear matrix inequalities for nonlinear control, a methodology that has become increasingly popular due to its systematicness and numerical implementability. Illustrations on common problems are made from existing literature: they are classified and discussed; advices are given to prevent them. Convex modelling is employed in linear parameter varying, Takagi-Sugeno models, and other convex structures in order to subsume or rewrite a nonlinear system for analysis or design via the direct Lyapunov method. Convexity plays a central role in allowing a finite set of vertex conditions in the form of linear matrix inequalities to be sufficient for the corresponding task. In contrast with other nonlinear methodologies, this one produces expressions resembling linear results, which makes it easier to grasp while often inducing subtle mistakes.

Keywords:
Linear Matrix Inequality, Direct Lyapunov Method, Convex Model, Nonlineal Control, Domain of Attraction.

Resumen

Este trabajo analiza los errores comunes al usar el modelado convexo y las desigualdades matriciales lineales para el control no lineal, una metodología que se ha vuelto cada vez más popular debido a su sistematicidad e implementabilidad numérica. Los ejemplos sobre problemas comunes son tomados de literatura existente: se clasifican, discuten y se dan consejos para prevenirlos. El modelado convexo se emplea en la variación de parámetros lineales, los modelos Takagi-Sugeno y otras estructuras convexas para subsumir o reescribir un sistema no lineal para el análisis o diseño a través del método directo de Lyapunov. La convexidad juega un papel central al permitir que un conjunto finito de condiciones en forma de desigualdades matriciales lineales sea suficiente para la tarea correspondiente. A diferencia de otras metodologías no lineales, ésta produce expresiones que se asemejan a resultados lineales, lo que hace que sea más fácil de comprender y, a menudo, induce errores sútiles.

Palabras Clave:
Desigualdades Matriciales Lineales, Método Directo de Lyapunov, Modelo Convexo, Control no Lineal, Dominio de atracción.

1. Preliminaries

Nonlinear control via convex structures (Lendek *et al.*, 2010) and linear matrix inequalities (LMIs) (Boyd *et al.*, 1994) have become a very well-established methodology in the last three decades; its origins can be traced back to *approximate* linear parameter varying (LPV) systems (Shamma, 2012), Takagi-Sugeno (TS) fuzzy models (Tanaka y Wang, 2001), and linear fractional transformations (LFT) (Cockburn y Morton, 1997). Since the appearance of the sector nonlinearity methodology in Taniguchi *et al.* (2001); Ohtake *et al.* (2003), which allows *rewriting* a nonlinear system as a convex sum of linear ones within a region of interest, the use of convex modelling and LMIs has become a legitimate nonlinear control tool, i.e., formally based on the direct Lyapunov method, results on analysis

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and design apply to the original nonlinear model (Bernal *et al.*, 2019).

Convex modelling can be briefly described as rewriting a nonlinear system

$$\dot{x}(t) = f(x) + g(x)u(t), \quad y(t) = s(x), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ gathers the system inputs, and $y \in \mathbb{R}^o$ represents measurable outputs, with $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $s : \mathbb{R}^n \to \mathbb{R}^o$ being sufficiently smooth vector fields of appropriate size, as a convex sum of linear systems within a region of interest $x \in \Omega \subset \mathbb{R}^n$, $0 \in \Omega$.

Such task begins by finding $A(x)$, $B(x)$, and $C(x)$ such that $f(x) = A(x)x$, $g(x) = B(x)$, and $s(x) = C(x)x$ with any non-constant term in $A(x)$, $B(x)$, and $C(x)$ being well-defined and bounded within Ω. If these conditions are met, the referred non-constant terms can be grouped in $z(x) = [z_1(x)\ z_2(x)\ \cdots\ z_p(x)]^T$ where $z_i(x) \in [z_i^0, z_i^1]$, $\forall x \in \Omega$. Based on these terms, the following can be defined:

$$w_0^i(x) = \frac{z_i^1 - z_i(x)}{z_i^1 - z_i^0},\ w_1^i(x) = 1 - w_0^i(x),\ i \in \{1, 2, \ldots, p\} \tag{2}$$

which in turn allows rewriting any non-constant term in $A(x)$, $B(x)$, and $C(x)$ as

$$z_i(x) = w_0^i(x)z_i^0 + w_1^i(x)z_i^1 = \sum_{i_j=0}^{1} w_{i_j}^i(x)z_i^{i_j},\ i \in \{1, 2, \ldots, p\}. \tag{3}$$

Convexity of functions $w_0^i(x)$, $w_1^i(x)$, $i \in \{1, 2, \ldots, p\}$ allows gathering them at the leftmost side of expressions (Bertsekas *et al.*, 2003), which means (1) is equivalent to:

$$\dot{x}(t) = \sum_{\mathbf{i}\in\mathbb{B}^p} \mathbf{w_i}(x)\left(A_\mathbf{i}x(t) + B_\mathbf{i}u(t)\right), \quad y(t) = \sum_{\mathbf{i}\in\mathbb{B}^p} \mathbf{w_i}(x)C_\mathbf{i}x(t), \tag{4}$$

where convexity lies on the fact that $\mathbf{w_i}(x) \in [0, 1]$, $\forall x \in \Omega$ and $\sum_{\mathbf{i}\in\mathbb{B}^p} \mathbf{w_i}(x) = 1$, $\forall x \in \mathbb{R}^n$; $\mathbb{B} = \{0, 1\}$, $\mathbf{i} = (i_1, i_2, \ldots, i_p)$, $\mathbf{w_i}(x) = w_{i_1}^1(x)w_{i_2}^2(x)\cdots w_{i_p}^p(x)$, $A_\mathbf{i} = A(x)|_{\mathbf{w_i}(x)=1}$, $B_\mathbf{i} = B(x)|_{\mathbf{w_i}(x)=1}$, $C_\mathbf{i} = C(x)|_{\mathbf{w_i}(x)=1}$.

The direct Lyapunov method (Khalil, 2002) comes at hand to perform stability analysis and/or controller/observer design for a nonlinear system that has been put in the convex form (4). Usually, a quadratic Lyapunov function candidate $V = x^T Px$, $P = P^T > 0$, is employed, along with controllers that employ the same nonlinearities of the system (parallel distributed compensation (PDC)) (Wang *et al.*, 1996) or observers that employ the measurable nonlinearities (Tanaka *et al.*, 1998). For illustration purposes, consider the stability theorem along with its proof, inspired in the classical result in Tanaka y Sugeno (1990):

Theorem 1. *The origin $x = 0$ of the nonlinear system* (1) *with $u = 0$ and equivalent convex model* (4) *$\forall x \in \Omega$, is asymptotically stable if there exists $P = P^T > 0$ such that $PA_\mathbf{i} + A_\mathbf{i}^T P < 0$, $\forall \mathbf{i} \in \mathbb{B}^p$. Any trajectory starting in $\{x : x^T Px \leq c\} \subset \Omega$ goes asymptotically to* 0.

Proof. Since $P = P^T > 0$, $V(x) = x^T Px$ is a valid Lyapunov function candidate; its time derivative is

$$\begin{aligned}\dot{V} &= 2x^T P\dot{x} = x^T P\left(\sum_{\mathbf{i}\in\mathbb{B}^p} \mathbf{w_i}(x)A_\mathbf{i}x\right) + \left(\sum_{\mathbf{i}\in\mathbb{B}^p} \mathbf{w_i}(x)A_\mathbf{i}x\right)^T Px \\ &= x^T \sum_{\mathbf{i}\in\mathbb{B}^p} \mathbf{w_i}(x)\left(PA_\mathbf{i} + A_\mathbf{i}^T P\right)x,\end{aligned}$$

which is negative-definite for $x \neq 0$ if $PA_\mathbf{i} + A_\mathbf{i}^T P < 0$, $\forall \mathbf{i} \in \mathbb{B}^p$, since $\mathbf{w_i}(x) \in [0, 1]$, $\forall x \in \Omega$; this makes $V(x)$ a Lyapunov function guaranteeing asymptotic stability of $x = 0$.

Since the validity of $V(x)$ as a Lyapunov function depends on $x \in \Omega$, it is clear that any trajectory within the Lyapunov level $\{x : x^T Px \leq c\} \subset \Omega$ goes asymptotically to 0. □

2. Boundedness and convexity of modelled nonlinearities

2.1. Incorrect handling of fractional convex expressions

As shown in Section 1, a convex model is not unique as there are many ways to factorize $f(x) = A(x)x$ and select non-constant terms within $A(x)$ (Sala *et al.*, 2005). Valid choices for non-constant expressions are *terms* and *factors*; *denominators* cannot be considered among these choices since we intend to use convexity to group terms at the rightmost side of the expressions while convex sums and convex functions $\mathbf{w_i}(x)$ are grouped at the leftmost side.

For example, consider expressions $z_1(x) \in [z_1^0, z_1^1]$ and $z_2(x) \in [z_2^0, z_2^1]$, $\forall x \in \Omega$, which can be rewritten as convex sums of their bounds $z_1(x) = \sum_{i_1=0}^{1} w_{i_1}^1(x)z_1^{i_1}$, $z_2(x) = \sum_{i_2=0}^{1} w_{i_2}^2(x)z_2^{i_2}$ with $w_0^j(x) + w_1^j(x) = 1$, $w_0^j, w_1^j \in [0, 1]$, $\forall x \in \Omega$, $j \in \{1, 2\}$; then

$$\begin{aligned}z_1(x) + z_2(x) &= \sum_{i_1=0}^{1} w_{i_1}^1(x)z_1^{i_1} + \sum_{i_2=0}^{1} w_{i_2}^2(x)z_2^{i_2} \\ &= \sum_{i_1=0}^{1}\sum_{i_2=0}^{1} w_{i_1}^1(x)w_{i_2}^2(x)\left(z_1^{i_1} + z_2^{i_2}\right), \\ z_1(x)z_2(x) &= \left(\sum_{i_1=0}^{1} w_{i_1}^1(x)z_1^{i_1}\right)\left(\sum_{i_2=0}^{1} w_{i_2}^2(x)z_2^{i_2}\right) \\ &= \sum_{i_1=0}^{1}\sum_{i_2=0}^{1} w_{i_1}^1(x)w_{i_2}^2(x)z_1^{i_1}z_2^{i_2},\end{aligned}$$

hold, but

$$\begin{aligned}z_1(x)/z_2(x) &= \left(\sum_{i_1=0}^{1} w_{i_1}^1(x)z_1^{i_1}\right)\Big/\left(\sum_{i_2=0}^{1} w_{i_2}^2(x)z_2^{i_2}\right) \\ &\neq \sum_{i_1=0}^{1}\sum_{i_2=0}^{1} w_{i_1}^1(x)w_{i_2}^2(x)z_1^{i_1}/z_2^{i_2},\end{aligned}$$

which unfortunately keeps appearing as an equivalence from time to time in a variety of proposals.

Conversely, lifting convex sums from their inverses is also, in general, incorrect. Indeed, while the implication

$$P_\mathbf{j}^{-1} > 0 \quad \Longrightarrow \quad \left(\sum_{\mathbf{j}\in\mathbb{B}^p} \mathbf{w_j}(x)P_\mathbf{j}\right)^{-1} > 0,$$

holds, it cannot be extended to other cases as

$$A_{\mathbf{i}}P_{\mathbf{j}}^{-1} > 0 \quad \not\Rightarrow \quad \sum_{\mathbf{i}\in\mathbb{B}^p} \mathbf{w}_{\mathbf{i}}(x)A_{\mathbf{i}}\left(\sum_{\mathbf{j}\in\mathbb{B}^p} \mathbf{w}_{\mathbf{j}}(x)P_{\mathbf{j}}\right)^{-1} > 0.$$

Such inverses of convex sums usually appear during the developments of the direct Lyapunov method (Daafouz y Bernussou, 2001) while applying matrix properties such as congruence, Schur complement, etcetera (Scherer, 2004).

Another source of problems during modelling of fractional expressions, is the fact that the resulting non-constant terms $z_i(x)$ must be bounded $\forall x \in \Omega$. This is a condition usually disregarded. For instance, in Chiu (2010), the non-constant term $z_1 = 1 - i_o/i_L$, where $i_L \in [-5\ 5]$ has been defined, but, since i_L can be 0, z_1 is not well defined in the specified compact set.

2.2. *Incorrect conclusions on nonlinear systems based on their convex approximations*

Convex models resulting from applying the methodology described in Section 1 are equivalent to their original nonlinear model (Taniguchi *et al.*, 2001; Sala y Ariño, 2009). Such equivalence allows drawing conclusions over the plant based on the convex representation and, ultimately, on the vertices of a simplex of linear systems. Nevertheless, when this equivalence is broken, conclusions cannot be drawn over the original nonlinear system based on *approximate* convex models, e.g., TS fuzzy models resulting from linearization at multiple operating points (Johansen *et al.*, 2000), LPV systems resulting from convex embedding of nonlinearities (Shamma, 2012), etc.

In the context of stability analysis, this means that asymptotic stability can be ensured of $x = 0$ for a convex model of the form (4) ($u = 0$) based on its vertices $A_{\mathbf{i}}$ if convexity holds, i.e., if $\exists\Omega$ such that $0 \in \Omega$ and $\mathbf{w}_{\mathbf{i}}(x) \in [0, 1]$, $\sum_{\mathbf{i}\in\mathbb{B}^p} \mathbf{w}_{\mathbf{i}}(x) = 1$. Additionally, if asymptotic stability of $x = 0$ has been ensured for a convex model of the form (4) ($u = 0$), the same thing can be claimed for the origin of the nonlinear model $\dot{x} = f(x)$ only if $f(x) = \sum_{\mathbf{i}\in\mathbb{B}^p} A_{\mathbf{i}}x$, i.e., if no approximation has been involved (Vázquez *et al.*, 2016). Moreover, any trajectory beginning in the outermost Lyapunov level within Ω is guaranteed to go asymptotically to 0 (see subsection 3.2).

Thus, while being negligible from a practical standpoint, drawing conclusions on a nonlinear plant based on convex approximations is not correct from a theoretical perspective, e.g., the PENDUBOT in Begovich *et al.* (2002) or the inverted pendulum in Meda-Campaña *et al.* (2017) have been successfully controlled or observed by means of an *approximate* fuzzy TS model, but this is a mere coincidence as no mathematical guarantees have been established.

2.3. *Incorrect assumptions on the membership functions*

In the LPV and TS fuzzy context, membership functions do not necessarily come from modelling nonlinear terms, i.e., their dependency is usually ignored in favour of global assumptions on their convexity Tanaka y Wang (2001); Tóth (2010). Nevertheless, when the sector nonlinearity approach described in Section 1 is employed to rewrite a nonlinear system as a convex model within a compact set of the states Ω, membership functions hold the convex sum property exclusively in Ω and their dependency on the state is not to be ignored Quintana *et al.* (2021).

Perhaps influenced by the LPV or fuzzy perspective, many works handle the membership functions without checking if their availability makes sense, leading to the following common problems:

1. Using membership functions that are not available: Observer design based on convex structures mimics parallel distributed compensation, which means that a nonlinear observer gain of the form

$$L(\hat{x}, y, u) = \sum_{\mathbf{k}\in\mathbb{B}^p} \mathbf{w}_{\mathbf{k}}(\hat{x}, y, u)L_{\mathbf{k}},$$

   where $\hat{x}$ is the state estimate, y is the output, and u is the input, is sought. Obviously, availability of the membership functions $\mathbf{w}_{\mathbf{k}}(\cdot)$ is crucial to make sense of such configuration; hence, the specified dependency on *available* variables. Nevertheless, such dependency is often ignored, leading to observers that use the very variables they are supposed to observe, e.g., in Chiu (2010) an observer uses membership functions that depend on $z_1 = 1 - i_o/i_L$ where $i_o = 0{,}5v_b$; but v_b is to be estimated by the observer, which is an inconsistency. It can be argued that the membership function might be available while some of the terms it depends upon are not; nevertheless, the mathematical relationship is *known* and, if solvable, the variables to be estimated by the observer can be directly obtained from this relationship.

2. Assuming availability of the membership functions, rendering an observer unnecessary: The simplest solution for observer design based on convex structures is to *assume* the membership functions are available, but even in this case consistency should be double-checked. For instance, consider (Lendek *et al.*, 2010, Example 4.1) where an observer is designed for a 2nd-order system whose output is $y = [x_1\ x_1x_2]^T$. Clearly, x_1 and x_2 can be directly calculated from the output y if $x_1 \neq 0$, i.e., there is no need of constructing an observer whose membership functions depend on the output.

3. Using membership functions that depend on uncertainties: Similar to the case where a controller/observer uses membership functions depending on unavailable variables, capturing uncertainties in membership functions makes them unavailable. Modelling stands, but unless a split of available/unavailable signals of membership functions is performed, the "captured" uncertainties should not be used due to their nature (Chadli y Guerra, 2012).

3. Connections with the direct Lyapunov method

3.1. *Incorrect use of time-invariant Lyapunov theorems for time-variant systems*

In the classical book Tanaka y Wang (2001), a TS model is defined as a "fuzzy blending of linear system models", where the "blending" is performed by membership functions which depend on "state variables, external disturbances, and/or time".

This class of systems is therefore not limited to time-invariant nonlinear ones. Yet, the Lyapunov theorems invoked along the book for analysis and design are those reserved for the time-invariant case, e.g., the stability of the origin of a TS system of the form $\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))A_i x(t)$, where $h_i(z(t))$ are convex functions depending on states, parameters, and/or time, is determined by means of a Lyapunov function $V(x) = x^T P x$, $P = P^T > 0$, such that $PA_i + A_i^T P < 0$, $i \in \{1, 2, \ldots, r\}$, regardless of the fact that such Lyapunov function candidate is only adequate for time-invariant systems.

This is not a technicality that can be ignored; take for instance the time-varying system $\dot{x} = A(t)x(t)$, $x \in \mathbb{R}^2$, in (Khalil, 2014, Example 4.22), with

$$A(t) = \begin{bmatrix} -1 + 1{,}5\cos^2 t & 1 - 1{,}5\sin t\cos t \\ -1 - 1{,}5\sin t\cos t & -1 + 1{,}5\sin^2 t \end{bmatrix}, \qquad (5)$$

where it has been proven that the states go to infinity no matter how close $x(0)$ is set from the origin, and regardless of the fact that $\sigma(A(t)) = -0{,}25 \pm 0{,}25\sqrt{7}j$. Nevertheless, once t is replaced by x_1 as to obtain $\dot{x} = A(x_1)x$, where $A(x_1)$ has obviously the same eigenvalues as $A(t)$, the origin becomes asymptotically stable. What does it mean for convex modelling and stability analysis via LMIs? It implies that any convex model of both systems with the same bounds may lead to the same vertices $A_{\mathbf{i}}$, based on which stability is determined; the only difference between both representations being the dependency of convex functions $\mathbf{w}_{\mathbf{i}}(\cdot)$ on time or states, i.e.:

$$\dot{x}(t) = \sum_{\mathbf{i}\in\mathbb{B}^p} \mathbf{w}_{\mathbf{i}}(t)A_{\mathbf{i}}x(t) \quad \text{time-variant,}$$

$$\dot{x}(t) = \sum_{\mathbf{i}\in\mathbb{B}^p} \mathbf{w}_{\mathbf{i}}(x_1)A_{\mathbf{i}}x(t) \quad \text{time-invariant.}$$

If $z_1 = \cos^2 t$ and $z_2 = \sin t\cos t$ for the time-variant system, or $z_1 = \cos^2 x_1$ and $z_2 = \sin x_1 \cos x_1$ for the time-invariant system, the vertices of the resulting convex model in both cases are

$$A_{00} = \begin{bmatrix} 0{,}44 & 1{,}15 \\ -0{,}85 & -0{,}94 \end{bmatrix}, \quad A_{01} = \begin{bmatrix} 0{,}44 & 0{,}7 \\ -1{,}29 & -0{,}94 \end{bmatrix},$$
$$A_{10} = \begin{bmatrix} 0{,}5 & 1{,}15 \\ -0{,}85 & -1 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} 0{,}5 & 0{,}7 \\ -1{,}29 & -1 \end{bmatrix},$$

provided that $-0{,}1 \le t \le 0{,}2$ (time-variant case) or $-0{,}1 \le x_1 \le 0{,}2$ (time-invariant case).

Besides the fact that bounding t is not a wise option when asymptotic stability is investigated, if a time-invariant version of the Lyapunov theorems is employed on the time-variant system[1], it may draw the incorrect conclusion that $x = 0$ is asymptotically stable. Indeed, the LMIs in Theorem 1, $P = P^T > 0$ and $PA_{\mathbf{i}} + A_{\mathbf{i}}^T P < 0$, $\mathbf{i} \in \mathbb{B}^2$, are feasible with

$$P = \begin{bmatrix} 628{,}53 & -481{,}52 \\ -481{,}52 & 720{,}75 \end{bmatrix}.$$

3.2. *Incorrect conclusions about the guaranteed domain of attraction*

When a nonlinear system is rewritten as a convex model following the sector nonlinearity approach sketched in Section 1, membership functions $\mathbf{w}_{\mathbf{i}}(x)$ are guaranteed to hold the convex sum property, i.e., $\sum_{\mathbf{i}\in\mathbb{B}^p} \mathbf{w}_{\mathbf{i}}(x) = 1$, $\mathbf{w}_{\mathbf{i}}(x) \in [0, 1]$, within $\Omega \subset \mathbb{R}^n$. As it has been seen in Section 1, convexity plays a central role in finding sufficient LMI conditions for analysis and/or design. If convexity is ignored, some properties still remain valid (though they are useless), for instance, $\sum_{\mathbf{i}\in\mathbb{B}^p} \mathbf{w}_{\mathbf{i}}(x) = 1$ and $\sum_{\mathbf{i}\in\mathbb{B}^p} \mathbf{w}_{\mathbf{i}}(x)A_{\mathbf{i}}x = f(x)$.

TS models in the fuzzy context did not consider these subtleties since the membership functions where *assumed* to hold the convex sum property *everywhere*, regardless of their specific dependency on time, states, or external variables. This is the reason why theorems in Tanaka y Wang (2001) and many other works within the TS fuzzy context, claim global stability of the origin, paying little or no attention to the actual bounds of the outermost Lyapunov level.

If a modelling region Ω is employed for sector nonlinearity approach, trajectories are guaranteed to behave as specified by the LMI analysis or design, only if they are contained in the outermost Lyapunov level within Ω, e.g., $\{x : x^T P x \le c, c > 0\} \subset \Omega$ (Bernal y Guerra, 2010; Lee y Kim, 2014; González *et al.*, 2017). Very often, careless simulations are presented where the trajectories are outside the outermost Lyapunov level within the modelling region or, plainly, outside the modelling region. See Figure 1 for an illustration of these remarks.

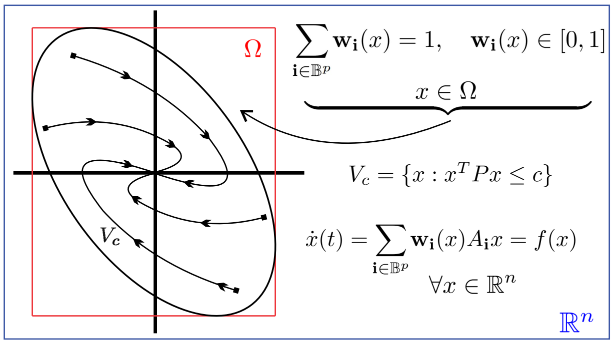

Figura 1: Outermost Lyapunov level within Ω

3.3. *Incorrect formulation of quadratic Lyapunov functions*

Since the quadratic Lyapunov function $V(\chi) = \chi^T P \chi$ is the simplest to accommodate when LMI tests are sought, and the developments usually involve a variety of matrix properties, it is common to find that inadequate or incorrect vectors have made their way into the final form of the time derivative $\dot{V}(\chi)$, leading to incorrect conclusions.

Indeed, consider the numerical examples in (Chiu y Ouyang, 2011, Section 5) where the maximum power point tracking (MPPT) is sought by means of a Lyapunov function of the form

$$V(x_c) = [x^T \; s \; \tilde{x}^T]P[x^T \; s \; \tilde{x}^T]^T,$$

with $x_c = [x^T \; s \; \tilde{x}^T]^T$. Since only $y = x_1$ (for maximum-power-voltage-based control) or $y = (G_a - n_p\gamma I_{rs}e^{\gamma x_1})x_1$ (for direct-maximum-power control) are required to go to 0 to achieve MPPT, guaranteeing $x_c \to 0$ by means of the LMI conditions in (Chiu y Ouyang, 2011, Theorem 1) goes far beyond

[1]For time-variant versions of the Lyapunov theorems, we refer the interested reader to Vidyasagar (2002).

the objective. This is clear from the fact that these conditions were supposedly used to obtain the controller/observer scheme for the simulation in (Chiu y Ouyang, 2011, Figure 4), where $x_1 = v_{pv} \to x_{1ss} > 10$; this is a contradiction as $x_c \to 0$ implies $x_1 \to 0$.

A similar misunderstanding can be found in the discrete-time context, where several works using Lyapunov functions that depend on past/future values of the state have been used, claiming bigger chances of feasibility or numerical advantages (Guerra *et al.*, 2012). Fundamentals on dynamical systems are very clear about the state being *all* the information required to predict the future (Kailath, 1980); at the same time, Lyapunov theory establishes asymptotic stability of the origin based on a Lyapunov function that depends *on the state vector* for time-invariant systems (Khalil, 2002). Based on these facts, it has been proven that the use of past/future values of the state adds no value to the feasibility chances of time-invariant systems, provided that enough computational resources are available (Ariño *et al.*, 2017).

4. Sufficiency and necessity of LMI conditions

As shown in Theorem 1 of Section 1, sufficient LMI conditions guaranteeing a nonlinear convex sum to be negative-definite are extracted from the vertices of the corresponding simplex. Conversely, it is necessary for *any* particular choice of values of the weighted convex sum to be a valid inequality; the same applies for properties such as controllability and observability: it is impossible for a set of sufficient controller/observer design LMIs to yield a feasible solution if *there exists* a particular combination of the convex sum that is not stabilizable/detectable (Meda-Campaña *et al.*, 2017).

To illustrate the previous points, consider the LMIs (28)-(30) in (Chiu, 2010, Theorem 2), which are

$$X_1, P_2 > 0, \tag{6}$$

$$\begin{bmatrix} \begin{pmatrix} A_{Ci}X_1 + X_1A_{Ci}^T + B_{ei}M_j \\ +M_j^T B_{ei}^T + \rho^2 H_s H_s^T \end{pmatrix} & X_1 Q_1 \\ Q_1 X_1 & X_1 \end{bmatrix} < 0, \tag{7}$$

$$\begin{bmatrix} A_i^T P_2 + P_2 A_i - E^T N_i^T - N_i E + Q_2 & P_2 H \\ H^T P_2 & \dfrac{1}{\rho^2} I_s \end{bmatrix} < 0, \tag{8}$$

where $X_1 \in \mathbb{R}^{4\times4}$, $P_2 \in \mathbb{R}^{3\times3}$, $N_i \in \mathbb{R}^{3\times2}$, $M_j \in \mathbb{R}^{1\times4}$ are decision variables; $A_{Ci} \in \mathbb{R}^{4\times4}$, $B_{ei} \in \mathbb{R}^{4\times1}$, $A_i \in \mathbb{R}^{3\times3}$, $E \in \mathbb{R}^{2\times3}$, $H \in \mathbb{R}^{3\times s}$ (where s is the rows number of the disturbance vector of the system) are known matrices corresponding to the TS exact modelling of the dc/dc buck converter given in Chiu (2010); $H_s = [H^T \ 0_{s\times1}]^T$; $Q_1 \in \mathbb{R}^{4\times4}$, $Q_2 \in \mathbb{R}^{3\times3}$, and $\rho \in \mathbb{R}$ are given for Lyapunov analysis. If feasible, controller and observer gains $K_i = M_i X_1^{-1}$ and $L_i = P_2^{-1} N_i$, $i, j \in \{1, 2, \ldots, 32\}$, can be found from LMIs (6)-(8).

Nevertheless, note that in order to satisfy LMIs (7), it is necessary to guarantee that all principal minors are negative-definite; thus, the following LMIs must hold

$$A_{Ci}X_1 + X_1A_{Ci}^T + B_{ei}M_j + M_j^T B_{ei}^T + \rho^2 H_s H_s^T < 0, \tag{9}$$

for $i, j \in \{1, 2, \ldots, 32\}$; applying the Schur complement to (9) yields

$$\begin{bmatrix} A_{Ci}X_1 + X_1A_{Ci}^T + B_{ei}M_j + M_j^T B_{ei}^T & H_s \\ H_s^T & -\dfrac{1}{\rho^2} I_s \end{bmatrix} < 0, \tag{10}$$

which, in turn, implies that

$$A_{Ci}X_1 + X_1A_{Ci}^T + B_{ei}M_j + M_j^T B_{ei}^T < 0$$

must be satisfied to guarantee LMIs (10) (and, consequently, (7)-(9)) hold.

Considering the values of A_{Ci} and B_{ei}, $i, j \in \{1, 2, \ldots, 32\}$, in Chiu (2010), it is possible to calculate the convex combinations $A_{conv} = \sum_{i=1}^{32} \mu_i A_{Ci}$, $B_{conv} = \sum_{i=1}^{32} \mu_i B_{ei}$, with the particular choice of values for μ_i, $\sum_{i=1}^{32} \mu_i = 1$, given in table 1:

$$A_{conv} = \begin{bmatrix} -1{,}026973/L & 0 & -1/L & 0 \\ 0 & 191{,}05 & 0 & 0 \\ 166{,}5 & 0 & 0 & 0 \\ 0 & -0{,}87055 & 0 & 0 \end{bmatrix}, B_{conv} = \begin{bmatrix} 14{,}73/L \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where $L = 150 \times 10^{-6}$H, given in Chiu (2010), has been written as a symbol to underline the fact that the validity of the argument that follows does not depend on any particular value of L. Importantly, the decimals in A_{conv} and B_{conv} are finite and complete.

Tabla 1: Values for μ_i

μ_i			
index	value	index	value
1	0.012	17	0.129
2	0.06	18	0.078
3	0.003	19	0.01
4	0.015	20	0.038
5	0.022	21	0.031
6	0.096	22	0.046
7	0.059	23	0.018
8	0.035	24	0.013
9	0.017	25	0.014
10	0.013	26	0.023
11	0.003	27	0.047
12	0.019	28	0.019
13	0.022	29	0.017
14	0.022	30	0.005
15	0.02	31	0.016
16	0.03	32	0.048

An L-dependent linear transformation T

$$T = \begin{bmatrix} -\dfrac{\sqrt{c_1 - c_2} + c_3}{c_4 L} & \dfrac{\sqrt{c_1 - c_2} - c_3}{c_4 L} & 0 & 0 \\ 0 & 0 & c_5 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

where parameters c_i, $i \in \{1, 2, 3, 4, 5\}$ are shown in table 2 (obtained with all their decimal positions via the Symbolic Toolbox in MATLAB™R2015a) is applied to the pair (A_{conv}, B_{conv})

in order to split the controllable and uncontrollable parts; the transformed matrices are:

$$\bar{A} = T^{-1}A_{conv}T = \begin{bmatrix} \bar{A}_{11} & 0_{2\times 2} \\ 0_{2\times 2} & \bar{A}_{22} \end{bmatrix}, \quad \bar{B} = T^{-1}B_{conv} = \begin{bmatrix} \bar{B}_1 \\ 0_{2\times 1} \end{bmatrix},$$

where $\bar{A}_{22} = \mathrm{diag}\{191{,}05, 0\}$ (regardless of the value of L); therefore, the pair (A_{conv}, B_{conv}) is not stabilizable. Consequently, by continuity arguments, there is a neighbourhood of values around those of the referred pair which yield an infinite number of convex combinations that are not stabilizable. This implies that LMIs (6)-(8) in (Chiu, 2010, Theorem 2) cannot be feasible, for any value of L; this is confirmed for $L = 150 \times 10^{-6}$H as in the example in Chiu (2010), both using the LMI Toolbox (Gahinet *et al.*, 1995) and Yalmip/SeDuMi (Sturm, 1999), even if H is sought.

Tabla 2: Values for parameters c_i

parameter	value
c_1	334239387371312443540948353025
c_2	211063824938000195349201083695104
c_3	578134402514945
c_4	187462334489296896
c_5	-3821000/17411

5. BMIs disguised as LMIs

LMIs belong to the class of convex optimization problems, which are guaranteed to be solved in polynomial time, where "being solved" means being able to find a solution if the problem is feasible or determine that no solution exists (unfeasible) (Boyd *et al.*, 1994). No ambiguity can prevail at this level. Least-squares and linear programming also belong to this class. In contrast, there is no certainty about the feasibility of a non-convex problem for which a method has been tried yielding no results; moreover, if a solution is found it might not be "optimal" in any sense (local minima) (VanAntwerp y Braatz, 2000). Common examples of non-convex problems are bilinear matrix inequalities (BMIs), which, due to the reasons above, are not the desired form for analysis and design of nonlinear systems via convex modelling (Kau *et al.*, 2007).

BMIs are not to be mistaken for LMIs that have not been properly worked out, e.g., when performing controller design, a middle step is finding $P = P^T > 0$ and F such that

$$P(A + BF) + (A + BF)^T P < 0,$$

which is not an LMI in P and F. Nevertheless, a *bijective* change of variables turns this inequality into an LMI because, in fact, the solution space for P and F above is indeed convex, namely,

$$AX + BM + XA^T + M^T B^T < 0,$$

where $X = P^{-1}$ and $M = FP^{-1}$. Conversely, sufficient LMI conditions can be found for solving a BMI problem if the solution space has been "cut" into a convex one.

Despite these facts, many works can be found (fortunately in receding numbers) whose design conditions are formulated as open or disguised BMIs (Cao *et al.*, 1998; Chadli y Guerra, 2012). It is understandable: finding sufficient LMI conditions for solving a specific problem can be very difficult; if found, these conditions can be prohibitively conservative (Crusius y Trofino, 1999).

A very good example of this situation is the development of piecewise controller design for TS systems: conditions in (Feng, 2003, Theorem 3.1) are what many refer to as *parameter-dependent* LMIs, an euphemism for the fact that conditions are BMIs with "only a handful" of scalar parameters to be "chosen", more specifically, scalars $\epsilon_l > 0$, $l \in \{1, 2, \ldots, m\}$ have to be chosen to solve the following "LMIs" for P_l and Q_l:

$$\begin{bmatrix} \Omega_l & P_l & Q_l^T \\ P_l & -M_{Pl}^{-1} & 0 \\ Q_l & 0 & -M_{Ql}^{-1} \end{bmatrix} < 0,$$

where $\Omega_l = P_l A_l^T + A_l P_l + Q_l^T B_l^T + B_l Q_l + \epsilon_l (E_{lA} E_{lA}^T + E_{lB} E_{lB}^T) + \gamma^{-2}(1 + \epsilon_l^{-1}) D_l D_l^T + \gamma^{-2}(1 + \epsilon_l) E_{lD} E_{lD}^T$, $M_{Pl} = \epsilon_l^{-1} I + (1 + 3\epsilon_l^{-1}) H_l^T H_l + (1 + 2\epsilon_l + \epsilon_l^{-1}) E_{lH}^T E_{lH}$, and $M_{Ql} = \epsilon_l^{-1} I + (1 + \epsilon_l + 2\epsilon_l^{-1}) G_l^T G_l + (1 + 3\epsilon_l) E_{lG}^T E_{lG}$.

These conditions were rapidly recognized as flawed and the authors took one step back to the BMI formulation, now openly exhibited in (Feng *et al.*, 2005, Theorem 3.1), where K_l and T in expressions such as $P_l = F_l^T T F_l$ and $\Omega_l = A_l^T P_l + P_l A_l + K_l^T B_l^T P_l + P_l B_l K_l + \epsilon_l E_{lA}^T E_{lA} + E_l^T W_l E_l + (1 + 3\epsilon_l^{-1}) H_l^T H_l + (1 + 3\epsilon_l) E_{lH}^T E_{lH}$ within an inequality, are sought. Notice that K_l and P_l appear multiplied in a couple of terms; hence, the BMI nature of the expression.

The mistakes of the first work were discussed in Shirani *et al.* (2010), but the BMI "correction" in Feng *et al.* (2005) is neither false nor useful as no method can decide in polynomial time the existence (or not) of a solution for a non-convex problem. Otherwise stated, if non-convex formulations were welcomed as "solutions" to control problems, practically all of them would have to be declared solved, even if no guaranteed method is offered to find a solution.

6. Conclusions

Common errors on using convex modelling and linear matrix inequalities for nonlinear control have been discussed. The main contributions of this work are listed below:

(A) Four problems in the area have been adressed.

(B) Both analytical and numerical problems have been illustrated.

(C) Clarifications concerning the difference between convex modelling, analysis and design for linear parameter varying and Takagi-Sugeno fuzzy models, on the one hand, and nonlinear systems on the other hand, have been made.

(D) Some advices to avoid the aforementioned issues have been included.

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