DESDE 2013
https://repository.uaeh.edu.mx/revistas/index.php/icbi/issue/archive
Pädi Boletín Científico de Ciencias Básicas e Ingenierías del ICBI

ISSN: 2007-6363

Publicación Semestral Pädi Vol. 11 No. Especial 3 (2023) 139-142

On the solution of linear functional two-term equations with shift
Solución de ecuaciones lineales funcionales con desplazamiento de dos términos

O. Karelin[a,*], A. Tarasenko[b]

[a] Área Académica de Ingeniería y Arquitectura, Universidad Autónoma del Estado de Hidalgo, 42184, Pachuca, Hidalgo, México.
[b] Área Académica de Matemáticas y Física, Universidad Autónoma del Estado de Hidalgo, 42184, Pachuca, Hidalgo, México.

Abstract

This work is dedicated to the study of linear functional equations with shift in Hölder space. Previously, for such operators, conditions for invertibility were found and the inverse operator was constructed by the authors. The operators are used in modelling systems with renewable resources. Here we propose another approach to solving functional equations with shift. With the help of an algorithm, the initial equation is reduced to the first iterated equations, then to the second iterated equation. Continuing this process, we obtain the *n*-th iterated equation and the limit iterated equation. We prove a theorem on the equivalence of the original and the iterated equations. Based on an analysis of the solvability of the limit equation, we find a solution to the original equation. The solution is the sum of an infinite product and a functional series. The results, and the methods for obtaining them, are transparent and not as cumbersome compared to previous works.

Keywords: Linear functional operator with shift, Non-Carlemann shift, Hölder space, Infinite product, Systems with renewable resources.

Resumen

Este trabajo está dedicado al estudio de ecuaciones lineales funcionales con desplazamiento en los espacios de Hölder. Previamente, para tales operadores, condiciones para la invertibilidad fueron hallados y el operador inverso fue construido por los autores. Los operadores se usan en el modelado de sistemas con recursos renovables. Aquí proponemos un enfoque diferente para la solución de ecuaciones funcionales con desplazamiento. Con la ayuda de un algoritmo, la ecuación inicial es reducida a la ecuación de primera iteración, después a la ecuación de segunda iteración. Continuando el proceso, obtenemos la ecuación de *n*-ésima iteración y a la ecuación iterada límite. Demostramos el teorema sobre la equivalencia de la ecuación original con las ecuaciones iteradas. Basándose en el análisis de la solubilidad de la ecuación límite, hallamos una solución a la ecuación original. La solución es la suma de un producto infinito y una serie funcional. Los resultados, y los métodos para obtenerlos, son transparentes y no tan complicados en comparación con trabajos anteriores.

Palabras Clave: Operador lineal funcional con desplazamiento, Desplazamiento no-Carlemann, Espacio de Hölder, Producto infinito, Sistemas con recursos renovables.

1. Introduction

We introduce the spaces of Hölder function $H_\mu(J)$, $0<\mu<1$, $J=[0,1]$ and the spaces of Hölder functions with weight $H_\mu^0(J,\rho)$, $\rho(x-0)^{\mu_0}(1-x)^{\mu_1}$, $\mu<\mu_0<1+\mu$, $\mu<\mu_1<1+\mu$.

In these spaces, we consider the functional equation with shift $\varphi(x)-b(x)\varphi[\alpha(x)]=g(x)$. We write this equation in operator form: $A\varphi(x)=g(x)$, $A=I+bB_\alpha$, where $B_\alpha\varphi(x)=\varphi[\alpha(x)]$. The coefficient-function $b(x)$ belongs to the space $H_\mu(J,\rho)$, the free member-function $g(x)$ belongs to the space $H_\mu(J)$ or to $H_\mu^0(J,\rho)$, and an unknown function $\varphi(x)$ is sought in $H_\mu(J)$ or in $H_\mu^0(J,\rho)$, respectively. The function $\alpha(x)$ is not arbitrary; certain strict requirements are imposed on it.

When we were modelling systems with renewable resources, Tarasenko, A., et al. (2010), Karelin, O., et al. (2015), Karelin, O., et al. (2019), Karelin, O., et al. (2018), we actively used functional operators with shift and related equations. Applications to economic and environmental problems were discussed Karelin, O., Tarasenko, A., et al. (2019).

*Autor para la correspondencia: karelin@uaeh.edu.mx
Correo electrónico: karelin@uaeh.edu.mx (Oleksandr Karelin), anataras@uaeh.edu.mx (Anna Tarasenko).

Historial del manuscrito: recibido el 14/08/2023, última versión-revisada recibida el 04/10/2023, aceptado el 17/10/2023, publicado el 20/11/2023. **DOI:** https://doi.org/10.29057/icbi.v11iEspecial3.11495

We formulate the result previously obtained by the authors Tarasenko, G., y Karelin, O. (2016), Karelin, O., et al. (2022), about the invertibility of operator A and the form of the inverse operator A^{-1} , in order to further compare it with our present results and the proposed approach.

In this work, we propose and develop another method for solving the equation $A\varphi(x) = g(x)$. The main idea of the proposed method is as follows: proceeding from to the original equation $A\varphi = g$, we construct the first iteration equation $\varphi = bB_\alpha bB_\alpha\varphi + bB_\alpha g$. Then, using the same algorithm, we construct the second iteration equation

$$\varphi = bB_\alpha bB_\alpha^2 bB_\alpha^2\varphi + bB_\alpha bB_\alpha^2 g + bB_\alpha g + g,\cdots,$$

construct the *n*-th iteration equation

$$\varphi = (bB_\alpha)^n\varphi + G_n, \quad G_n = (bB_\alpha)^n g + \cdots + bB_\alpha g + g.$$

Note the important relation$(bB_\alpha)^n = b\cdot(B_\alpha b)\ldots(B_\alpha^{n-1}b)B_\alpha^n$.

Passing to the limit, we get

$$\varphi(x) = \lim_{n\to\infty}[\Omega^{n-1}B^n\varphi(x) + G_n(x)],$$

where $\Omega_{n-1} = (b)(B_\alpha b)\ldots(B_\alpha^{n-1}b)$.

We prove the theorem on equivalence of the original and the iterated equations. Requirements were imposed on the function $\alpha(x)$ so that $\lim\limits_{n\to\infty} B_\alpha^n\varphi(x) = \varphi(1)$ for$x \neq 0$. Based on the analysis of solvability of the limit equation, we find a solution to the original equation which amounted to the content of the second theorem. The solution of the equation is expressed in terms of a convergent infinite product and a convergent functional series. The results obtained and the method developed are transparent and less cumbersome compared to previous works Tarasenko, G., y Karelin, O. (2016), Karelin, O., et al. (2022). The proposed approach will be used to study more complex equations such as three-term equations and some non-linear equations Tarasenko, A., et al. (2023), Khinchin, A. Yu. (1964). This approach is interesting in that it can be applied to solving equations that arise in developing models of renewable systems with organisms in different states Tarasenko, A., Karelin, O., et al. (2021), for example, being sick, being healthy without immunity, being healthy with immunity and being vaccinated.

2. Two-term linear equation with shift. Iterated equations.

Let us recall the definitions of the Hölder space and of the Hölder space with weight, since the equations that we study will be considered precisely in these spaces. A function $\varphi(x)$ that satisfies the following condition on $J = [0,1]$,

$$|\varphi(x_1) - \varphi(x_2)| \leq C|x_1 - x_2|^\mu, \quad x_1 \in J, x_2 \in J, \mu \in (0,1),$$

is called a Hölder function with exponent μ and constant C. Hölder functions form a space $H_\mu(J)$. The norm in $H_\mu(J)$ is defined by

$$\|f(x)\|_{H_\mu(J)} = \|f(x)\|_C + \|f(x)\|_\mu,$$

where

$$\|f(x)\|_C = \max_{x\in J}|f(x)|, \ \|f(x)\|_\mu = \sup_{x_1,x_2\in J, x_1\neq x_2}\frac{|f(x_1)-f(x_2)|}{|x_1-x_2|^\mu}.$$

Let ρ be a power function, which has zeros at the endpoints $x = 0$, $x = 1$: $\rho(x) = (x-0)^{\mu_0}(1-x)^{\mu_1}$, $\mu < \mu_0 < 1+\mu$, $\mu < \mu_1 < 1+\mu$.

The functions that become Hölder functions have zero values at the points $x = 0, x = 1$, after being multiplied by $\rho(x)$, form a Banach space $H_\mu^0(J,\rho)$. The norm in the space $H_\mu^0(J,\rho)$ is defined by

$$\|f(x)\|_{H_\mu^0(J,\rho)} = \|\rho(x)f(x)\|_{H_\mu(J)}.$$

Let $\alpha(x)$ be a bijective orientation-preserving shift on J: if $x_1 < x_2$, then $\alpha(x_1) < \alpha(x_2)$ and $\alpha(x)$ has only two fixed points $\alpha(0) = 0$, $\alpha(1) = 1$ and $\alpha(x) \neq x$,when $x \neq 0$, $x \neq 1$. In addition, let $\alpha(x)$ be a differentiable function with $\frac{d}{dx}\alpha(x) \neq 0$ and $\frac{d}{dx}\alpha(x) \in H_\mu(J)$. We will assume that these requirements are always fulfilled. An example of such shift is $\alpha(x) = \frac{x^2+x}{2}$. The shift operator is definde by the formula

$$(B_\alpha\varphi)(x) = \varphi[\alpha(x)].$$

Consider the operator $A = aI - bB_\alpha$ with coefficients from the Hölder space $a \in H_\mu(J), b \in H_\mu(J)$. Operator A acts on Hölder space with weight$H_\mu^0(J,\rho)$.

We will now formulate the conditions of invertibility for operator A in the space of Hölder class functions with weight [6, 7].

Operator A, acting in Banach space $H_\mu^0(J,\rho)$ is invertible if the following condition is fulfilled: $\theta_\alpha[a(x), b(x), H_\mu^0(J,\rho)] \neq 0$, $x \in J$, where the function σ_α is defined by

$$\theta_\alpha[a(x), b(x), H_\mu^0(J,\rho)] = \begin{cases} a(x), when\ |a(0)| > [\alpha'(0)]^{-\mu_0+\mu}|b(0)|\ and \\ \qquad |a(1)| > [\alpha'(1)]^{-\mu_0+\mu}|b(1)|; \\ b(x), when\ |a(0)| < [\alpha'(0)]^{-\mu_0+\mu}|b(0)|\ and \\ \qquad |a(1)| < [\alpha'(1)]^{-\mu_0+\mu}|b(1)|; \\ 0\ in\ other\ cases. \end{cases}$$

In this article, we propose a different approach to the solution of the equation $a(x)\varphi(x) - b(x)\varphi[\alpha(x)] = g(x)$ in terms of convergent infinity product and convergent series based on the recurrence relation. For simplicity, we assume that $a(x) = 1$.

In the space $H_\mu^0(J,\rho)(H_\mu(J))$, consider the equation:

$$\varphi(x) - b(x)\varphi[\alpha(x)] = g(x). \tag{1}$$

Writing the equation in operator form, we obtain $A\varphi(x) = g(x)$ or $\varphi(x) - b(x)B_\alpha\varphi(x) = g(x)$. Coefficient $b(x)$ belongs to $H_\mu(J)$, the known free term on the right side $g(x)$ belongs to $H_\mu^0(J,\rho)(H_\mu(J))$, the unknown function φ is sought in space $H_\mu^0(J,\rho)(H_\mu(J))$.

Operators A and B_α are linear and act in a Banach space $H_\mu^0(J,\rho)(H_\mu(J))$.

We represent the equation in the recurrent form:

$$\varphi(x) = b(x)B_\alpha\varphi(x) + g(x). \quad (2)$$

Substituting the expression for the unknown function, taken from equation (2), into the right side of the same equation, we get an equation after the first iteration. Let us denote the obtained equation as the first iterated equation. Hence, the statement follows:

$$\varphi(x) = (b(x)B_\alpha)[b(x)B_\alpha\varphi(x) + g(x)] + g(x), \quad (3)$$
$$\varphi(x) = (bB_\alpha)(bB_\alpha)\varphi + (bB_\alpha)g + g.$$

We note that operator B_α has the multiplicative property

$$B_\alpha(\varsigma(x)\cdot\xi(x)) = (B_\alpha\varsigma(x))\cdot(B_\alpha\xi(x)). \quad (4)$$

Using the multiplicative property (4), we write $B_\alpha bI = (b(x))B_\alpha I$ and operator B_α in (3) can be moved to the end of the expression:

$$\varphi = (b)\cdot(B_\alpha b)B_\alpha\varphi + b\cdot B_\alpha g + g,$$
$$\varphi = (b)\cdot(B_\alpha b)\cdot(B_\alpha^2 b)\cdot B_\alpha^2\varphi + b\cdot(B_\alpha b)B_\alpha^2 g + bB_\alpha g + g.$$

Here, we have indicated the results of the first and the second iteration. Continuing the iterative process, at the step n the solution $\varphi(x)$ takes the form:

$$\varphi(x) = (bB_\alpha)^n\varphi(x) + G_n(x), \quad (5)$$

where $\quad G_n = (bB_\alpha)^n g(x) + \cdots + bB_\alpha g(x) + g(x).$

3. Equivalence of the original equation and the iterated equations. Solution of two-term linear equations with shift.

Let us establish connections between the original and the iterated equations. Here, for convenience, we rewrite the original equation in varios forms:

$$\varphi - bB_\alpha\varphi = g;\ (I - bB_\alpha)\varphi = g;\ \varphi = bB_\alpha\varphi + g;\ A\varphi = g,$$

where $A = I + bB_\alpha$.
We add the term $-bB_\alpha\varphi$ to the right and the left sides of the iterated equation (3) and slightly transform it. We obtain an equation that is equivalent to the iterated equation:

$$\varphi - bB_\alpha\varphi = bB(bB_\alpha\varphi + g) + g - bB_\alpha\varphi,$$
$$\varphi - bB_\alpha\varphi - g = bB(bB_\alpha\varphi - \varphi + g),$$

and, finally, we get the iterated equation containing inside the initial equation

$$(I + bB_\alpha)(\varphi - bB_\alpha\varphi - g) = 0. \quad (6)$$

We start with the following statements.

The set of solutions of the original equation (1) is included in the set of solutions of the iterated equation (3); if the iterated equation (3) is unsolvable, then the original equation (1) is also unsolvable. However, it does not follow from the equality (6) that the solutions of the iterated equation are the solutions of the original equation. To prove the equivalence of the equations, additional reasoning is required.

Theorem 1. *The original equation and all iterated equations are equivalent that is if the* n-*th iterated equation obtained at the n step of the iterative process, has a solution, then the original equation also has the same solution.*

Proof. For each solution $\psi(x)$ of the n-th iterated equation, the identity holds:

$$\psi(x) \equiv (bB_\alpha)^n\psi(x) + G_n(x), \quad (7)$$

The same function $\psi(x)$ will be the solution to the *(n+1)*-th iterated equation:

$$\psi \equiv (bB_\alpha)^{n+1}\psi + G_{n+1} \text{ or } \psi \equiv (bB_\alpha)[(bB_\alpha)^n\psi + G_n] + g$$

Expression in square brackets $(bB_\alpha)^n\psi + G_n$ is equals to function $\psi(x)$ according to identity (7). We substitute and we have $\psi(x) \equiv b(x)B_\alpha\psi(x) + g(x)$. What was required to prove. Q.E.D. ■

Now, we will calculate the initial values $\varphi(1)$ and $\varphi(0)$. These values will be useful to us in further study of the initial equation (1) and the construction of its solutions. At $x = 1$, we obtain the relation

$$\varphi(1) - b(1)(B_\alpha\varphi)(1) = g(1).$$

Since the constant does not change under the action of the shift operator, $B_\alpha\varphi(1) = \varphi(1)$, we have

$$\varphi(1) - b(1)\varphi(1) = g(1),\ \ (1 - b(1))\varphi(1) = g(1),$$

$$\varphi(1) = \frac{g(1)}{1-b(1)},\ \ 1 - b(1) \neq 0.$$

Similarly, finding $\varphi(0)$ is reduced to a calculation by the formula

$$\varphi(0) = \frac{g(0)}{1-b(0)},\ \ 1 - b(0) \neq 0.$$

Let us return to the original equation (1) in its reduced form (2). In section 2, at step n of the iterative process, we construct the n-th iterated equation (5),

$$\varphi(x) = (bB_\alpha)^n + G_n \quad \text{or} \quad \varphi = \Omega_{n-1}B_\alpha^n\varphi + G_n,$$

where

$$\Omega_{n-1} = b(B_\alpha b)\cdots(B_\alpha^{n-1}b),\quad G_n = g + bB_\alpha g + \cdots (bB_\alpha)^n g,$$

which is equivalent to the original equation $\varphi = bB_\alpha\varphi + g$. Note that $(bB_\alpha)^n$ is the producto of the operators, and Ω_{n-1} is the product of the functions.

According to Theorem 1, in every step of the iterative process we get an equation equivalent to the original equation. Passing to the limit

$$\varphi(x) = \lim_{n\to\infty}[\Omega^{n-1}B^n\varphi(x) + G_n(x)],$$

we get a theorem:

Theorem 2. *If the infinite product converges to some function* $\Omega(x) = \lim_{n\to\infty}\Omega_n(x)$ *, and the functional series converges to some function* $G(x) = \lim_{n\to\infty}G_n(x)$ *and* $b(1) \neq 1$*, then the original equation* $\varphi(x) = bB_\alpha\,\varphi(x) + g(x)$ *is uniquely solvable and this solution is determined by the formula*

$$\varphi(x) = \Omega(x)\frac{g(1)}{1-b(1)} + G(x).$$

References

Tarasenko, A., Karelin, O., Pérez-Lechuga, P. and Gonzalez-Hernandez, M. (2010). Modelling systems with renewable resources based on functional operators with shift. Applied Mathematics and Computation, 216(7), 1938-1944. DOI: 10.1016/j.amc.2010.03.023

Karelin, O., Gonzalez-Hernandez, M., Hernandez-Romero, N. and Seck Tuoh-Mora, J. (2015). Study of renewable systems based on functional operators with shift. Applied Mathematical Sciences, 9(144), 7167-7173. DOI: 10.12988/ams.2015.510622

Karelin, O., Tarasenko, A., Barabash, O., Gonzalez-Hernandez, M. and Medina-Marin, J. (2019). Study of Changes of Individual Parameter of Resources in the Modelling of Renewable Systems. In: Ao S., Gelman, L., Kim, H. (Ed.), Transactions on Engineering Technologies, Springer Nature Singapore, 19-28.

Karelin, O., Tarasenko, A., Zolotov, V. Gonzalez-Hernandez, M. (2018). Study of the equilibrium state of systems with two renewable resources based on functional operators with shift. IAENG International Journal of Applied Mathematics, 48(1), 98-104.

Karelin, O., Tarasenko A., Zolotov V., Gonzalez-Hernandez, M., (2019) Mathematical Models for the Study of Resource Systems Based on Functional Operators with Shift, Transactions on Engineering Technologies, Ao, S.-I., et al. (eds), Springer Nature, 95-108.

Tarasenko, G., Karelin, O. (2016) Condition of Invertibility for Functional Operators with Shift in Weighted Hölder spaces. Ukrainian Mathematical Journal, 67(11), 1758-1769.

Karelin, O., Tarasenko, A., Gonzalez-Hernandez, M. (2022). On invertibility of some functional operators with shift. Applied Mathematics, 13(8), 651-657. DOI: 10.4236/am.2022.138040

Tarasenko, A., Karelin, O., Gonzalez-Hernandez, M., Karelina, D. (2023). On the Solution of Equations with Linear-Fractional Shifts. WSEAS Transaction on Mathematics, 22, 254-258. DOI: 10.37394/23206.2023.22.30

Khinchin, A. Yu. (1964). Continued Fraction, University of Chicago Press.

Tarasenko, A., Karelin, O., Gonzalez-Hernandez, M., Barabash, O. (2021). Modelling Systems with Elements in Several States, WSEAS Transaction on Environment and Development, Vol. 17, 244-252. DOI: 10.37394/232015.2021.17.25