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Symmetric texture-zero mass matrices with positive eigenvalues
Matrices simétricas de masa del tipo texturas con eigenvalores positivos

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Abstract

Within the context of the texture-zeros mechanism for fermionic mass matrices, we provide necessary and sufficient conditions for the characteristic polynomial coefficients such that it has real, simple and positive roots. We translate these conditions in terms of invariants from congruent matrices. Then, all symmetric texture-zero matrices are counted and classified. We apply the result from the first part to analyze the three, two and one zero texture matrices in a systematic way. Finally, we solve analytically the V_{CKM} mixing matrix for the four zero sets; we also analyze the $|V_{CKM}|$ for a particular case of four zeros, four zero-perturbed and three zero sets.

Keywords: Textures, V_{CKM} matrix, parallel structure, non parallel structure.

Resumen

En el contexto del mecanismo de texturas con ceros para las matrices de masa fermiónicas, proporcionamos condiciones necesarias y suficientes sobre los coeficientes del polinomio característico tal que tenga raíces reales, simples y positivas. Traducimos estas condiciones en términos de invariantes de matrices congruentes. Entonces, todas las matrices simétricas de texturas con ceros son contadas y clasificadas. Aplicamos el resultado de la primera parte para analizar de manera sistemática las matrices de texturas con uno, dos y tres ceros. Finalmente encontramos analíticamente la matriz de mezcla V_{CKM} para los conjuntos de cuatro ceros; también analizamos la $|V_{CKM}|$ para un caso particular de cuatro ceros, cuatro de ceros perturbados y tres de ceros.

Palabras Clave: Texturas, matriz V_{CKM}, estructura paralela, estructura no paralela.

1. Introduction

In the Standard Model (SM) with $SU(2) \times U(1)$ as the gauge group of electroweak interactions (Weinberg (1967)), the masses of quarks and charged leptons are contained in the Yukawa Sector. After Spontaneous Symmetry Breaking (SSB), the mass matrix is defined as:

$$M_f = \frac{v}{\sqrt{2}} Y_f, \qquad (f = u, d, l),$$

where v is the vacuum expectation value of the Higgs field and Y_f are the 3×3 Yukawa matrices, without loss of generality, we can consider them as Hermitian. The physical masses of the particles are closely related to mass matrix eigenvalues M_f, so, as a starting point, both must be real numbers. Mathematically speaking, a Hermitian mass matrix always guarantees that the eigenvalues are real, i.e., positive, negative or zero. To have a correct identification between eigenvalues and physical masses, the physical masses m_i are defined as the absolute values of the mass matrices eigenvalues $m_i = |\lambda_i|$. The following cases exist:

Positive eigenvalues.
We have a straight identification. Physical particle masses are mass matrices eigenvalues.

Negative eigenvalues.
In this case we have, $\lambda_i = -m_i$, and negative sign can be removed with an extra rotation.

Within the SM context, the mass matrix is unknown, the only trail of the quarks mass matrices is the V_{CKM} matrix, which is built by the product of left matrices that diagonalize the u and d-quark mass matrix (Kobayashi and Maskawa (1973)).

In 1977 Harald Fritzsch proposed a phenomenological study of mass matrices (Fritzsch (1977a)), the so-called texture-zeros mechanism [1], that consists of looking for the simplest

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[1] For excellent reviews see (Fritzsch and zhong Xing (2000); Fritzsch and Xing (1998)), and references there in.

pattern of mass matrices, which can result in a self-consistent way and it reproduces the V_{CKM} parameters obtained experimentally.

The aim of the paper is to establish a context where zero texture matrices have positive and different eigenvalues, that is, there is a natural relationship between physical particle masses and mass matrices eigenvalues. From all possible texture-zero structures (hermitian, non-hermitian as triangular matrices (Haussling and Scheck (1998), Kuo et al. (1999))), as our first proposal, we restrict our study to symmetric matrices, in order to simplify calculations and to explore if this idea can be reliable in the texture formalism.

Mathematically speaking, a symmetric texture matrix always guarantees real eigenvalues. However, any symmetric matrix does not fulfill the positivity condition for the eigenvalues. Moreover, a positive definite matrix has real and positive eigenvalues that are not necessarily different.

The organization of this paper is as follows. In Sec. II, we show analytically how the mass matrices appears in the SM context. In Sec. III, we find necessary and sufficient conditions on the characteristic polynomial coefficients such that their roots are real, simple and positive quantities. These conditions are rewritten in terms of the invariants of the congruent matrices, *i.e.*, trace, determinant and trace of the power matrix. In Sec. IV, we develop a simple notation that counts and classifies the texture-zero matrices, and we show that all symmetric matrices of 3×3 can be grouped into one-zero, two-zero and three-zero texture, in order to complete the counting of the matrix without zeros is included. In Sec. V, we apply systematically, the results of Sec. III to all matrices of the Sec. IV, and we show what kind of texture matrices have real, different and positive eigenvalues. Finally, in Sec. VI, we derive analytical expressions for all the V_{ckm} elements arising from the four-zero sets, then by choosing a particular case of a four-zero set, we compute the V_{CKM} matrix, and we perturb this case in order to improve the expressions for the V_{CKM} elements, finally, we took this case to the three-zero sets.

2. Preliminaries

In the Yukawa sector of the SM, the mass terms for quarks and charged leptons can be expressed as

$$\bar{u}_L M_u u_R + \bar{d}_L M_d d_R + \bar{l}_L M_l l_R, \tag{1}$$

where $u_{L(R)}$, $d_{L(R)}$ and $l_{L(R)}$ are the left(right)-handed quark and charged leptons fields for the u-sector (u, c, t), d-sector (d, s, b) and charged leptons (e, μ, τ) respectively. M_u, M_d and M_l are the mass matrices. Expressing the above equation in terms of the physical fields, one diagonalizes the mass matrices by unitary transformations

$$\begin{aligned} \bar{M}_u &= U_u^\dagger M_u U_u = Diag\left[\lambda_u, \lambda_c, \lambda_t\right], \\ \bar{M}_d &= U_d^\dagger M_d U_d = Diag\left[\lambda_d, \lambda_s, \lambda_b\right], \\ \bar{M}_l &= U_l^\dagger M_l U_l = Diag\left[\lambda_e, \lambda_\mu, \lambda_\tau\right], \end{aligned} \tag{2}$$

where U_f $(f = u, d, l)$ are in general complex unitary matrices. The quantities $\lambda_u, \lambda_d, \ldots$ etc. denote the eigenvalues of the mass matrices, i.e., the physical quark masses are $m_f = |\lambda_f|$

Re-expressing Eq. (1) in terms of physical fermion fields (f') as

$$\bar{u'}\bar{M}_u u' + \bar{d'}\bar{M}_d d' + \bar{l'}\bar{M}_l l', \tag{3}$$

where $\bar{f'} = \bar{f}\,U_f$ and $f' = U_f^\dagger f$, $(f' = u', d', l')$.
Eq.(2) implies that $\bar{M}_f$ and M_f, $(f = u, d, l)$ are congruent matrices, the relation of congruence is an equivalence relation, which implies a space partition into cosets. Any two elements that belong in the same coset have the following invariants: determinant, trace, trace of the power matrix, characteristic polynomial and their eigenvalues.[2] (Friedberg et al. (2006)).
Considering M_f as a 3×3 symmetric matrix with real coefficients, $\bar{M}_f$ is built as a diagonal matrix where its elements are the eigenvalues of M_f, these eigenvalues are found as the roots of its characteristic polynomial. In the following section, we give conditions on the coefficients of the characteristic polynomial from M_f, *i.e.*, on the M_f elements, such that this polynomial has three real, positive and simple roots.

3. Main Theorem

As was mentioned before, the physical quark masses are defined as the positive eigenvalues of the mass matrix, from a mathematical point of view, to obtain the eigenvalues it is necessary to compute the characteristic equation and its positive roots are the quark masses. In this section we present the conditions over the characteristic polynomial coefficients such that the polynomial characteristic roots are real, positive and different. We translate these conditions in terms of invariants of congruent matrices as trace and determinant of the mass matrix.

Theorem 1. *The polynomial of degree 3,* $p(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$, *has three different, real and positive roots if and only if the following conditions over its coefficients* a_0, a_1 a_2 *hold.*

1. $a_0, a_2 < 0 < a_1$.
2. $3a_1 < a_2^2$.
3. *If* $\lambda_4 = \dfrac{-a_2 + \sqrt{a_2^2 - 3a_1}}{3}$ *and* $\lambda_5 = \dfrac{-a_2 - \sqrt{a_2^2 - 3a_1}}{3}$, *then* $p(\lambda_4) < 0$ *and* $p(\lambda_5) > 0$.

Proof. See Appendix. ■

We observe that in the condition 3, λ_4 and λ_5 are the roots of the first derivative of $p(\lambda)$, and therefore the condition 2 implies that λ_4 and λ_5 are real numbers, in other words, $p(\lambda)$ has two critical points, this fact joins to the condition 1 implies that $0 < \lambda_5 < \lambda_4$.

The condition 3 ($p(\lambda_4) < 0$ and $p(\lambda_5) > 0$), means that the maximum value is positive, and the minimum value is negative, and therefore $p(\lambda)$ has three real and different roots. This condition can be replaced by

$$-2(a_2^2 - 3a_1)^{3/2} < 2a_2^3 - 9a_1a_2 + 27a_0 < 2(a_2^2 - 3a_1)^{3/2}, \tag{4}$$

[2] In this work, we will denote the product (TrA)(TrA) as Tr^2A. In the general case $(\mathrm{Tr}A)^n = \mathrm{Tr}^nA$ for n positive integer.

the first inequality is obtained by solving $p(\lambda_5) > 0$ and the second one is obtained by solving $p(\lambda_4) < 0$. The condition (4) can be rewritten as

$$|2a_2^3 - 9a_1a_2 + 27a_0| < 2(a_2^2 - 3a_1)^{3/2}. \tag{5}$$

It is convenient to rewrite the theorem 1 in terms of the invariants of congruent matrices. This creates directly a link between the matrix elements and its eigenvalues, which facilitates subsequent computations and applications. To implement this fact, first we write the coefficients of its characteristic polynomial $p(\lambda)$ in terms of its trace (TrM), trace of the square matrix (TrM^2) and its determinant ($\det M$) in the following form:

$$p(\lambda) = \lambda^3 - \mathrm{Tr}M\lambda^2 + \frac{1}{2}\left[\mathrm{Tr}^2M - \mathrm{Tr}M^2\right]\lambda - \det M. \tag{6}$$

Now we are ready to present the main theorem of this section

Theorem 2. *A real, symmetric matrix M has real, positive and different eigenvalues if and only if the following three conditions hold.*

1. (a) $\det M > 0$,
   (b) $\mathrm{Tr}M > 0$,
   (c) $\mathrm{Tr}M^2 < \mathrm{Tr}^2M$.
2. $\mathrm{Tr}^2M < 3\mathrm{Tr}M^2$.
3. $\left|\mathrm{Tr}M(5\mathrm{Tr}^2M - 9\mathrm{Tr}M^2) - 54\det M\right| < \sqrt{2}(3\mathrm{Tr}M^2 - \mathrm{Tr}^2M)^{3/2}$.

The theorem 2 will be applied to texture-zero matrices.

4. Texture-zero

A texture-zero matrix is a 3×3 matrix with zeros in some entries, the way to count them is the following: a zero in the main diagonal add as 1, while zero off main diagonal add as 1/2. We need to sum all zeros for both mass matrices u-quarks and d-quarks. For example, given M_u and M_d as

$$M_u = \begin{pmatrix} * & 0 & * \\ 0 & * & * \\ * & * & 0 \end{pmatrix}, \qquad M_d = \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix}. \tag{7}$$

For M_u we have one zero in the main diagonal, we add (+1) and 2 zeros off main diagonal that add $1(= 1/2 + 1/2)$, then M_u has a two-zero texture structure. Considering now M_d we have a three-zero texture structure $(1 + 2)$. Then, this set of matrices is said to have a five-zero texture structure.

We say: a parallel structure for M_u and M_d mass matrices means that if M_u has zeros in some places, then M_d has zeros in the same position than M_u. Non-parallel structure is when M_u and M_d not have the same parallel structure.

4.1. Notation

We start writing a symmetric matrix M in the form:

$$M = \begin{pmatrix} E & D & F \\ D & C & B \\ F & B & A \end{pmatrix}. \tag{8}$$

This matrix is well determined by specifying six capital letters (A, B, C, D, E, F) and their corresponding positions, then we introduce the following notation:

- $M(x)$ is a matrix with a zero in the capital letter x, $(x = A, B, C, D, E, F)$.
- $M(x, y)$ is a matrix with zeros in the capital letters x and y, $(x, y = A, B, C, D, E, F;\ x \neq y)$.
- $M(x, y, z)$ is a matrix with zeros in the capital letters x, y and z, $(x, y, z = A, B, C, D, E, F;\ x \neq y \neq z)$.

For example, a matrix with a zero in position F is:

$$M(F) = \begin{pmatrix} E & D & 0 \\ D & C & B \\ 0 & B & A \end{pmatrix}, \tag{9}$$

a matrix with zeros in the positions C and D is,

$$M(C, D) = \begin{pmatrix} E & 0 & F \\ 0 & 0 & B \\ F & B & A \end{pmatrix}, \tag{10}$$

finally, a matrix with zeros in the positions C, D and F is,

$$M(C, D, F) = \begin{pmatrix} E & 0 & 0 \\ 0 & 0 & B \\ 0 & B & A \end{pmatrix}. \tag{11}$$

Using this notation, we are able to list all possible textures.
One-zero texture structure.
We have 6 different matrices, which are:
$M(A)$, $M(C)$, $M(E)$, $M(B)$, $M(D)$, $M(F)$.
Two-zero texture structure.
In this case, we have 15 possibilities, which are:
$M(A, E)$, $M(A, C)$, $M(C, E)$,
$M(A, B)$, $M(A, D)$, $M(A, F)$,
$M(B, C)$, $M(C, D)$, $M(C, F)$,
$M(B, E)$, $M(D, E)$, $M(E, F)$,
$M(B, F)$, $M(B, D)$, $M(D, F)$.
Three-zero texture structure.
For this case, there are 20 different matrices, which are:
$M(A, B, C)$, $M(A, C, F)$, $M(A, C, D)$,
$M(A, B, E)$, $M(A, E, F)$, $M(A, D, E)$,
$M(B, C, E)$, $M(C, E, F)$, $M(C, D, E)$,
$M(A, B, F)$, $M(A, B, D)$, $M(A, D, F)$,
$M(B, C, F)$, $M(B, C, D)$, $M(C, D, F)$,
$M(B, E, F)$, $M(B, D, E)$, $M(D, E, F)$,
$M(A, C, E)$, $M(B, D, F)$.

Now we are ready to analyze which kinds of textures have three different and positive eigenvalues, applying in each case one of the theorems presented in previous sections.

5. Combined analysis

The aim of this section is give to quark mass matrices the structure of zero textures and find which of these structures have real, positive and different eigenvalues. To start the analysis systematically, we need to implement another sub-classification, which depends on whether the matrix has or not zeros in the main diagonal, doing this, first we analyze the three-zeros textures, after this, we study the two-zero textures and finally the 1-zeros textures.

5.1. Three-zero analysis

According to the sub-classification given above, the tree-zero textures present the following cases:

1. Without zeros in the main diagonal, there is one case $M(B,D,F)$.
2. With one zero in the main diagonal exist 9 cases: $M(A,B,F)$, $M(A,B,D)$, $M(A,D,F)$, $M(B,C,F)$, $M(B,C,D)$, $M(C,D,F)$, $M(B,E,F)$, $M(B,D,E)$, $M(D,E,F)$.
3. With two zeros in the main diagonal there are 9 cases: $M(A,B,C)$, $M(A,C,F)$, $M(A,C,D)$, $M(A,B,E)$, $M(A,E,F)$, $M(A,D,E)$, $M(B,C,E)$, $M(C,E,F)$, $M(C,D,E)$.
4. With three zeros in the main diagonal we have only 1 case ($M(A,C,E)$).

We obtain a total of 20 different possibilities. We only present the analysis of the following three cases.

- Applying the Theorem 2 (1b) the trivial $M(A,C,E)$ case is ruled out[3]

- Now, we analyze the Fritzsch six-zero texture given by $M(C,E,F)$ (Fritzsch (1977b)). Applying again the Theorem 2 (1b) we must have $\text{Tr}M(C,E,F) = A > 0$, from the condition (1a) $\det M(C,E,F) = -D^2A < 0$ that is a contradiction, because of that, this six-zero texture is ruled out under this context.

- Next, we analyze the following texture $M(A,D,F)$. The condition (1b) of the Theorem 2 we have that $\text{Tr}M(A,D,F) = C + E > 0$ and from (1a) $\det M(A,D,F) = -EB^2 > 0$, $\Leftrightarrow E < 0 \Rightarrow C > 0 \Rightarrow EC < 0$. The condition (1c) of the Theorem 2 implies that $0 < E^2 + C^2 + 2B^2 < E^2 + C^2 + 2EC \Rightarrow 0 < EC$ and we have a contradiction and this texture is also ruled out.

We have analyzed the other 17 cases, and we found that the only case that is not excluded is $M(B,D,F)$, obviously being A, B and C the eigenvalues ($A \neq C \neq E > 0$).

5.2. Two-zero analysis

These kind of textures have the following cases:

1. Without zeros in the main diagonal there are 3 cases: $M(B,F)$, $M(B,D)$, $M(D,F)$.
2. With one zero in the main diagonal exist 9 cases: $M(A,B)$, $M(A,D)$, $M(A,F)$, $M(B,C)$, $M(C,D)$, $M(C,F)$, $M(B,E)$, $M(D,E)$, $M(E,F)$
3. With two zeros in the main diagonal there are 3 cases: $M(A,E)$, $M(A,C)$, $M(C,E)$.

We present the analysis of some more representative cases:

- We start with the matrix $M(C,E)$. If we compute $\text{Tr}^2 M(C,E)$, $\text{Tr}M(C,E)^2$ and we apply the condition (1c) of the Theorem 2, we obtain:
$$2(D^2 + F^2 + B^2) + A^2 < A^2, \tag{12}$$
that is a contradiction. We have found that $M(A,E)$ and $M(A,C)$ are ruled out too.

- The second example is the Fritzsch four-zero texture given by $M(E,F)$ (Fritzsch and zhong Xing (2003)). From the Theorem 2 follows that the condition (1a) $\det M(E,F) = -AD^2 > 0$ implies $A < 0$, and of the condition (1b) $\text{Tr}M(E,F) = C + A > 0$ we have that $C > 0$ and then $AC < 0$. Now we compute $\text{Tr}^2 M(C,E)$, $\text{Tr}M(C,E)^2$ and using the condition (1c) of the Theorem 2, we obtain:
$$2(D^2 + B^2) + C^2 + A^2 < C^2 + A^2 + 2AC, \tag{13}$$
then $AC > 0$, that is a contradiction.

We have analyzed the eight cases $M(A,B)$, $M(A,D)$, $M(A,F)$, $M(B,C)$, $M(C,D)$, $M(C,F)$, $M(B,E)$, $M(D,E)$ and we found that are ruled out.

The cases that are in agreement with the condition (1) of the Theorem 2 are $M(B,F)$, $M(B,D)$ and $M(D,F)$; this means that it exist a range of values of (B,F), (B,D) and (D,F) where these textures have real, positive and different eigenvalues.

5.3. One-zero analysis

Here we only have two cases,

1. Without zeros in the main diagonal, three different possibilities are $M(B)$, $M(D)$ and $M(F)$.
2. With one zero in the main diagonal, there are also three different possibilities: $M(A)$, $M(C)$ and $M(E)$.

We only present the analysis of $M(A)$. The condition (1b) produces $E + C > 0$, the condition (1a) implies that $2BDF - B^2E - F^2C > 0$ and the condition (1c) gives $2(B^2 + D^2 + F^2) + E^2 + C^2 < E^2 + C^2 + 2EC$, the last three inequalities are equivalent with
$$E + C > 0, \tag{14}$$
$$2BDF > B^2E + F^2C, \tag{15}$$
$$0 < B^2 + D^2 + F^2 < EC, \tag{16}$$
from Eq. (14) and (16) we have that $E > 0$ and $C > 0$, therefore
$$-2BF\sqrt{EC} < B^2E + F^2C, \tag{17}$$
$$2BF\sqrt{EC} < B^2E + F^2C, \tag{18}$$
now if $BF > 0$, the inequalities (16, 18, 15) produce the following chain of inequalities
$$2BF\sqrt{B^2 + D^2 + F^2} < 2BF\sqrt{EC} < B^2E + F^2C < 2BDF,$$
and then
$$\sqrt{B^2 + D^2 + F^2} < D,$$
that is a contradiction. If $BF < 0$ we use Eq.(17). We have analyzed the other 2 cases $M(C)$, $M(E)$ and we found that are ruled out.

[3]In this work, we are looking for textures with positive and different eigenvalues. Therefore, textures with two equal eigenvalues or one of them negative, we say that they are ruled out.

The cases that are in agreement with the condition (1) of the Theorem 2 are $M(B)$, $M(D)$ and $M(F)$.

Summing up this section, the zero texture mass matrices that have real, positive and different eigenvalues are:

$$M(B,\ F),\ M(B,\ D),\ M(D,\ F),\ M(B),\ M(D)\ \text{and}\ M(F).$$

Our results are in agreement with (Branco et al. (2000)), where the authors using Weak Basic Transformations, they have shown that any symmetric texture with (1,1) zero entry has at least one negative eigenvalue.

6. V_{CKM} Properties

Another important quantity that any quark mass matrices need to satisfy is reproduces the experimental values of the V_{CKM}. For this reason, in this section we analyze the V_{CKM} phenomenology, in the first part and considering a set of four zeros for mass matrices, we note the presence of zeros in the V_{CKM} that depend on if we have a parallel and non-parallel structures in the quark mass matrices, in the second part, we choose a particular non-parallel case and compute the V_{CKM} matrix. In order to fit this V_{CKM} matrix with the experimental V_{CKM} matrix we introduce a perturbation analysis. Finally, we present a set of three zeros where the V_{CKM} fits numerically.

6.1. V_{CKM} from 4-zero texture set

In the previous sections, it was shown that $M(B,F)$, $M(B,D)$ and $M(D,F)$ are matrices with simple, real and different eigenvalues. When the mass matrix of u-type quarks and the mass matrix of and d-type quarks both have a parallel structure (*e.g.* $M_u = M_u(B_u, F_u)$ and $M_d = M_d(B_d, F_d)$), one direct implication is that the V_{CKM} has the same texture structure as the mass matrices ($V_{CKM} = V_{CKM}(B_{CKM}, F_{CKM})$) and we cannot reproduce the experimental values of the V_{CKM} elements because of that, all these three cases are ruled out.

Now, if the mass matrix of u-type quarks and the mass matrix of and d-type quarks have not a parallel structure, all nine cases were analyzed and always we find one zero element (off main diagonal) in the V_{CKM} matrix. We present the case where the best fit of the V_{CKM} is found, this is because we can obtain analytic expressions as well as a lot of information about the mass matrices. For this, we choose the mass matrix $M(D,F)$ texture for u-type quarks, and the matrix $M(B,F)$ texture for d-type quarks. Then we have that

$$M_u = \begin{pmatrix} m_u & 0 & 0 \\ 0 & C_u & B \\ 0 & B & A_u \end{pmatrix}, \qquad M_d = \begin{pmatrix} E_d & D & 0 \\ D & C_d & 0 \\ 0 & 0 & m_b \end{pmatrix}. \tag{19}$$

From the appendix (B.5) and (B.6), the above matrices take the form:

$$M_u = \begin{pmatrix} m_u & 0 & 0 \\ 0 & \mu_{ct} + \sqrt{\delta_{tc}^2 - B^2} & B \\ 0 & B & \mu_{ct} - \sqrt{\delta_{tc}^2 - B^2} \end{pmatrix}, \tag{20}$$

$$M_d = \begin{pmatrix} \mu_{ds} + \sqrt{\delta_{sd}^2 - D^2} & D & 0 \\ D & \mu_{ds} - \sqrt{\delta_{sd}^2 - D^2} & 0 \\ 0 & 0 & m_b \end{pmatrix}, \tag{21}$$

where $\mu_{qiqj} = \frac{m_{qi} + m_{qj}}{2}$ and $\delta_{qiqj} = \frac{m_{qi} - m_{qj}}{2}$ (with $m_{qi} > m_{qj}$, $i,\ j = 1,2,3$ and $q = u,\ d$). The quantities μ_{qiqj} and δ_{qiqj} have a interesting physical meaning, the first one is the average mass, and for the second one, we can be rewriting as $2\delta_{qiqj} + m_{qj} = m_{qi}$, then $2\delta_{qiqj}$ is the quantity that distinguishes the masses, i.e. the particles m_{qi} and m_{qj} are different because their mass are different and the factor of difference is $2\delta_{qiqj}$. The matrices that diagonalize the mass matrices are

$$U_u = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix}, \qquad U_d = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{22}$$

where $\sin\alpha = \frac{D}{\sqrt{D^2+(y_d-m_d)^2}}$ and $\sin\beta = \frac{B}{\sqrt{B^2+(y_u-m_c)^2}}$.

Now we computing the $V_{CKM} = U_u^T U_d$ matrix

$$V_{CKM} = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\cos\beta\sin\alpha & \cos\beta\cos\alpha & -\sin\beta \\ -\sin\beta\sin\alpha & \sin\beta\cos\alpha & \cos\beta \end{pmatrix}. \tag{23}$$

Setting:

$$\sin\alpha = V_{us} = \lambda, \qquad \sin\beta = -V_{cb} = -A\lambda^2, \tag{24}$$

where λ is the Wolfenstein parameter and A is a real number of order one.

The V_{CKM} matrix takes the form:

$$V_{CKM} = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & 0 \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3 & -A\lambda^2 & 1 \end{pmatrix} + O(\lambda^4). \tag{25}$$

With this election of texture structure of the mass matrices of quarks, we can reproduce (in Wolfenstein parametrization) eight V_{CKM} parameters, and the V_{ub} element is zero. Now, with this information, we can know explicitly each element of the mass matrices, from Eq.(24) we have that

$$\sin\alpha = \frac{D}{\sqrt{D^2 + (y_d - m_d)^2}} = V_{us}, \tag{26}$$

$$\sin\beta = \frac{B}{\sqrt{B^2 + (y_u - m_c)^2}} = -V_{cb}, \tag{27}$$

the solutions for D and B are:

$$D_0 = \pm 2\delta_{sd} V_{us}\sqrt{1 - V_{us}^2} \approx \pm 2\delta_{sd} V_{us}, \tag{28}$$

$$B_0 = \pm 2\delta_{tc} V_{cb}\sqrt{1 - V_{cb}^2} \approx \pm 2\delta_{tc} V_{cb}, \tag{29}$$

and the mass matrices are:

$$M_u = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_c + 2\delta_{tc}V_{cb}^2 & \pm 2\delta_{tc}V_{cb} \\ 0 & \pm 2\delta_{tc}V_{cb} & m_t - 2\delta_{tc}V_{cb}^2 \end{pmatrix}, \tag{30}$$

$$M_d = \begin{pmatrix} m_d + 2\delta_{sd}V_{us}^2 & \pm 2\delta_{sd}V_{us} & 0 \\ \pm 2\delta_{sd}V_{us} & m_s - 2\delta_{sd}V_{us}^2 & 0 \\ 0 & 0 & m_b \end{pmatrix}. \tag{31}$$

Finally the mass matrices can be written as:

$$M_u = \bar{M}_u + 2\delta_{tc} V_{cb}^2 \Delta M_u \pm 2\delta_{tc} V_{cb} \delta M_u, \quad (32)$$
$$M_d = \bar{M}_d + 2\delta_{sd} V_{us}^2 \Delta M_d \pm 2\delta_{sd} V_{us} \delta M_d, \quad (33)$$

where the matrices ΔM_u, ΔM_d, δM_u and δM_d are given by:

$$\Delta M_u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \Delta M_d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\delta M_u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \delta M_d = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (34)$$

We observe that the mass matrices have three contributions; the first one ($\bar{M}$) comes from a diagonal matrix, where its elements correspond to mass quarks, the second contribution (ΔM) is a correction of diagonal entries, and it is characterized by the square of V_{us} and V_{cb} elements respectively. The last contribution (δM) is off-diagonal correction characterized by a lineal contribution of V_{us} and V_{cb}. Note that: off diagonal contribution is bigger than the diagonal ones.

From Eq.(23) we can see that, we get one zero in V_{ub} element, the experimental value for this element is $0.00351^{+0.00015}_{-0.00014}$ (Olive et al. (2014)), this invite us to apply perturbation theory to 4-zero texture in order to get a better numerical approximation for this V_{CKM} element.

6.1.1. Perturbative analysis of four-zero texture set

As was mentioned in the previous section, when we consider a four-zero texture set as a structure of mass matrices of quarks, the presence of zeros in the V_{CKM} matrix is unavoidable, the aim of this part of the paper is use perturbation theory to remove these zeros and get small quantities.

We consider that quark mass matrices can be divided into two parts:

$$M_q = M_{q(2T)} + \epsilon N_q, \quad (35)$$

where $M_{q(2T)}$ is a two-zero texture, N_q is known as perturbation matrix and ϵ_q a small parameter (See appendix for more details). The new contributions to V_{CKM} matrix comes from an antisymmetric matrix X_q.

In the example presented before, where $M_u = M_u(D_u, F_u)$ and $M_d = M_d(B_d, F_d)$ are the structures for the quark mass matrices, one can reproduce eight experimental values of V_{CKM} elements and one get that the V_{ub} element is zero. To remove this zero, first we consider a perturbation on $M_u = M_u(D_u, F_u)$ and keeping $M_d = M_d(B_d, F_d)$ unchanged, after that, we will interchange the roles.

We consider that, M'_u mass matrix differs a small quantity [4] ϵ from M_u in the positions $(1,2)$, $(2,1)$, $(1,3)$ and $(3,1)$.

$$M'_u = \begin{pmatrix} m_u & \epsilon\, a_u & \epsilon\, b_u \\ \epsilon\, a_u & C_u & B \\ \epsilon\, b_u & B & A_u \end{pmatrix}, \quad (36)$$

[4] $|\epsilon\, a_u| \sim |\epsilon\, b_u| \ll m_u$, $|C_u|$, $|B_u|$ and $|A_u|$.

where ϵ is a real parameter in the interval $0 \leq \epsilon \leq 1$ and a_u, b_u are new free parameters with mass units. M'_u matrix can be written in the form

$$M'_u = M_u(D_u, F_u) + \epsilon\, N_u, \quad (37)$$

where $M_u(D_u, F_u)$ matrix is given in (30) and the perturbation matrix N_u is given by

$$N_u = \begin{pmatrix} 0 & a_u & b_u \\ a_u & 0 & 0 \\ b_u & 0 & 0 \end{pmatrix}. \quad (38)$$

Following the analysis given in the appendix and applying right perturbation at first order in ϵ, we find

$$O_u = U_u(1 + \epsilon X_u), \quad (39)$$

where the U_u matrix is given in Eq.(22) and X_u matrix is:

$$X_u = \begin{pmatrix} 0 & x_{1u} & x_{2u} \\ -x_{1u} & 0 & x_{3u} \\ -x_{2u} & -x_{3u} & 0 \end{pmatrix}, \quad (40)$$

and its elements are: $x_{1u} = \dfrac{a_u \cos\beta}{m_c - m_u} - \dfrac{b_u \sin\beta}{m_c - m_u}$, $x_{2u} = \dfrac{a_u \sin\beta}{m_t - m_u} + \dfrac{b_u \cos\beta}{m_t - m_u}$, and $x_{3u} = 0$.

The new V'_{CKM} matrix takes the following form:

$$V'_{CKM} = O_u^T U_d, \quad (41)$$
$$= (1 - \epsilon X_u) U_u^T U_d, \quad (42)$$
$$= (1 - \epsilon X_u) V_{CKM}. \quad (43)$$

After some algebra, using Eq.(27) and Eq.(40), we get that, the new element V'_{ub} has the form:

$$V'_{ub} = \left(\frac{V_{cb}^2}{m_c - m_u} - \frac{1 - V_{cb}^2}{m_t - m_u} \right) \epsilon b_u - \left(\frac{1}{m_c - m_u} + \frac{1}{m_t - m_u} \right) V_{cb} \sqrt{1 - V_{cb}^2}\, \epsilon a_u. \quad (44)$$

The biggest contribution comes from:

$$V'_{ub} = \left(\frac{V_{cb}^2}{m_c} - \frac{1}{m_t} \right) \epsilon b_u - \left(\frac{V_{cb}}{m_c} \right) \epsilon a_u. \quad (45)$$

We have a non-zero element, which its magnitude depends on V_{cb}, m_c, m_t and perturbation parameters ϵb_u and ϵa_u. The numerical contribution from coefficient of term ϵb_u goes like 10^{-6}, while the numerical contribution from coefficient of term ϵa_u goes like 10^{-5}. The smallest numerical element of $M_u(D_u, F_u)$ matrix is m_u, then we consider that, the maximum value of perturbation parameters ϵb_u and ϵa_u is $m_u/10$. We scanned all allowed range of $\epsilon\, a_u$ and $\epsilon\, b_u$ parameters, and we get that the best numerical absolute value is 8×10^{-6}. The absolute values of new V'_{CKM} elements are:

$$|V'_{CKM}| = \begin{pmatrix} 0.9753 & 0.2208 & 8 \times 10^{-6} \\ 0.2206 & 0.9745 & 0.039 \\ 0.0086 & 0.0380 & 0.9992 \end{pmatrix}. \quad (46)$$

The values of the mass matrix parameters of M'_u were: $|\epsilon\, a_u| = |\epsilon\, b_u| = 0.2 \sim \frac{|m_u|}{10} \ll, m_u = 2.3$, $|A_u| = 172739$, $|C_u| =$

1531.2, $|D| = 6697.47$, all quantities in MeV. The numerical values that corresponding to second order in ϵ are $O(10^{-8})$ or less. For left and left-right perturbations (See Appendix), the numerical values were the same order.

Now we consider that M'_d mass matrix differs a small quantity [5] $\epsilon\, a_d$, $\epsilon\, b_d$ from M_d in the positions $(1,3)$, $(3,1)$, $(3,2)$ and $(2,3)$.

$$M'_d = \begin{pmatrix} E_d & D_0 & \epsilon\, a_d \\ D_0 & C_d & \epsilon\, b_d \\ \epsilon\, a_d & \epsilon\, b_d & m_b \end{pmatrix}, \qquad (47)$$

M'_d matrix can be written in the form

$$M'_d = M_d(B_d, F_d) + \epsilon\, N_d,$$

where $M_d(B_d, F_d)$ matrix is given in Eq.(30) and N_d matrix is given by

$$N_d = \begin{pmatrix} 0 & 0 & a_d \\ 0 & 0 & b_d \\ a_d & b_d & 0 \end{pmatrix}. \qquad (48)$$

Applying right perturbation at first order in ϵ, we find

$$O_d = U_d(1 + \epsilon X_d), \qquad (49)$$

where the matrix X_d is:

$$X_d = \begin{pmatrix} 0 & x_{1d} & x_{2d} \\ -x_{1d} & 0 & x_{3d} \\ -x_{2d} & -x_{3d} & 0 \end{pmatrix}, \qquad (50)$$

and its elements are $x_{1d} = 0$, $x_{2d} = \dfrac{a_d \cos\alpha}{m_b - m_d} - \dfrac{b_d \sin\alpha}{m_b - m_d}$ and $x_{3d} = \dfrac{a_d \sin\alpha}{m_b - m_s} + \dfrac{b_d \cos\alpha}{m_b - m_s}$.

The new V'_{CKM} matrix takes the following form:

$$\begin{aligned} V'_{CKM} &= V_u^T\, O_d, & (51)\\ &= V_u^T\, V_d(1 + \epsilon X_d), & (52)\\ &= V_{CKM}(1 + \epsilon X_d). & (53) \end{aligned}$$

After some algebra, using Eq.(26) and Eq.(50), we get that the biggest contribution to new element V'_{ub} has the form:

$$V'_{ub} = V_{us}\left(\frac{m_s}{m_b^2}\right)\epsilon b_d + \left(\frac{1}{m_b}\right)\epsilon a_d. \qquad (54)$$

We have a non-zero element, which its magnitude depends on V_{us}, m_s, m_b and perturbation parameters $\epsilon\, a_d$ and $\epsilon\, b_d$. The numerical contribution from the coefficient of term b_d goes like 10^{-6}, while the numerical contribution from coefficient of term a_d goes like 10^{-4}. The smallest numerical element from matrix $M_d(B_d, F_d)$ is E_d, then we consider that the maximum value of perturbation parameters $\epsilon\, a_d$ and $\epsilon\, b_d$ is $E_d/10$. We scanned all allowed range of $\epsilon\, a_d$ and $\epsilon\, b_d$ parameters, and we get that, the best numerical absolute value is 2×10^{-4}. The absolute values of the new V'_{CKM} elements are:

$$|V_{CKM}| = \begin{pmatrix} 0.9742 & 0.2253 & 0.0002 \\ 0.2251 & 0.9734 & 0.0406 \\ 0.0089 & 0.0396 & 0.9991 \end{pmatrix}. \qquad (55)$$

[5] $|\epsilon\, a_d| \sim |\epsilon\, b_d| \ll |E_d|, |C_d|, |D|, m_b$

The values of the parameters were: $|\epsilon\, a_d| = |\epsilon\, b_d| = 0.9 \sim \frac{|E_d|}{10} \ll |E_d| = 9.37$, $|C_d| = 90.42$, $|D| = 20.32$, $m_b = 4180$, all quantities in MeV. The numerical values that correspond to second order in ϵ are $O(10^{-8})$ or less. For left and left-right perturbations, the numerical values were the same order of magnitude.

Also, we have numerically analyzed all possibilities to get a perturbation on both mass matrices without get better numerically values in the V_{CKM} matrix.

From the analysis of this section, we conclude that four-zero texture set in the normal and perturbed cases are ruled out, because they can not reproduce the experimental values of the V_{CKM} matrix.

6.2. V_{CKM} *from three-zero texture set*

The next case of structure is a three-zero texture set, which is born when one type of quarks has as mass matrix $M(B, F)$, $M(B, D)$ or $M(D, F)$ and the other type of quarks has mass matrix $M(B)$, $M(D)$ or $M(F)$. We have, in total, 18 possible combinations.

From the analysis presented before, we can point out two issues:

- We can introduce a V_{ub} element different from zero, setting appropriately, the values (1,3) and (3,1) in M_d matrix. From (54), we can note a lineal dependence between V_{ub} and the perturbation, ϵa_d and if $|\epsilon a_d| \sim E_d$ we obtain the numerical value of V_{ub} very close that the experimental one. Then we will consider that F_d is the same order as E_d.
- The quark mass matrix can be split in two parts, a diagonal part plus off-diagonal contributions, which both of them are in power series of V_{us} and V_{cb} elements.

Considering the above statements, we take $M(D, F)$ as a 2-zero structure for u-type quarks, *i.e* it has the form given in (32) and the matrix that diagonalize it is (22). For d-quarks, we take $M(B)$ as 1-zero structure given by

$$M_d = \begin{pmatrix} E_d & D_d & F_d \\ D_d & C_d & 0 \\ F_d & 0 & A_d \end{pmatrix}, \qquad (56)$$

where each element is parameterized as:

main diagonal elements
$A_d = m_b + x\, V_{us}^3$,
$C_d = m_s - 2\delta_{sd} V_{us}^2 + y\, V_{us}^3$,
$E_d = m_d + 2\delta_{sd} V_{us}^2 + z\, V_{us}^3$

off diagonal elements
$D_d = +2\delta_{sd} V_{us}$,
$F_d = m_d + 2\delta_{sd} V_{us}^2$,

where (x, y, z) are variables to find. Now as M_d is congruent with $Diag\,[m_d,\, m_s,\, m_b]$ we can write the following equations:

$$\begin{aligned} \mathrm{Tr}\, M_d &= m_d + m_s + m_b, \\ det\, M_d &= m_d\, m_s\, m_b, \qquad (57)\\ \frac{1}{2}\left[\mathrm{Tr}^2 M_d - \mathrm{Tr} M_d^2\right] &= m_d\, m_s + m_d\, m_b + m_s\, m_b, \end{aligned}$$

this set of equations has six solutions for (x, y, z), and we choose the solution that $E_d < C_d < A_d$ is hold, i.e. the numerical values for (x, y, z) are $(1.84199,\ x+z,\ -1.84417)$, then numerically the matrix M_d results:

$$M_d = \begin{pmatrix} 9.42827 & 19.802896 & 9.380186 \\ 19.802896 & 90.1575 & 0 \\ 9.380186 & 0 & 4180.21 \end{pmatrix}, \quad (58)$$

and the numerical absolute values of V_{CKM} elements are:

$$|V_{ckm}| = \begin{pmatrix} 0.974118 & 0.226027 & 0.00224906 \\ 0.225925 & 0.973273 & 0.0412108 \\ 0.00712578 & 0.0406523 & 0.999148 \end{pmatrix}, \quad (59)$$

that is in agreement with the experimental value of V_{CKM} matrix.

This is a good example that shows that three-zero texture sets are viable candidates to model the quark mass matrices.

7. Conclusions

In this article, assuming that, by definition, the physical mass of quarks and charged leptons are the positive eigenvalues of the mass matrices. We found the necessary and sufficient conditions over the characteristic polynomial coefficients from any symmetric 3×3 matrix, so that it has real, simple and positive roots. We apply this formalism to analyze the symmetric texture-zero quark matrices, from all set of matrices, only the following structures $M(B, F)$, $M(B, D)$, $M(D, F)$, $M(B)$, $M(D)$ and $M(F)$ are in agree with this condition. In the texture-zero formalism, the matrices have variable coefficients, the conditions 2 and 3 impose restrictions over these coefficients, this means that, we need to find the complete domain of the coefficients in both mass matrices, u-type quarks and d-type quarks, in order to approximate the experimental values of the V_{CKM} matrix, we develop analytically the case of four zero sets, and we show the set that gives the best approximation to the V_{CKM} matrix and always a zero element in the theoretical V_{CKM} matrix is found, to remove this zero, we implement a perturbation method, and we analyze the four zero texture set, even with these results the four-zero texture sets cannot reproduce the experimental value of V_{ub}. The quark mass matrix can be split into two parts, a diagonal part plus off-diagonal contributions, which both of them are in power series of V_{CKM} elements, statement that is valid for three and four zero texture sets. With an example, we show that three-zero texture sets are viable to model the quark mass matrices, and this structure is minimal, which means that they have real, positive and different eigenvalues, and also it reproduces the experimental values of the V_{CKM} matrix.

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Appendix A. Proof Theorem

In this appendix, we proof the theorem 1. For this we need the following statement:

Lemma 1. *The polynomial of second degree $p(\lambda) = \lambda^2 + a_1\lambda + a_0$ has two real, simple and positive roots if and only if the following condition holds:*

$$a_1 < 0 < a_0 < \frac{a_1^2}{4}. \quad (A.1)$$

Proof. Follows from a simple computation. ■

Now we remember and proof the theorem 1.

Theorem 1 The polynomial of degree 3, $p(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$ has three different, real and positive roots if and only if the following conditions over its coefficients a_0, a_1 a_2 hold.

1. $a_0, a_2 < 0 < a_1$.
2. $3a_1 < a_2^2$.
3. If $\lambda_4 = \dfrac{-a_2 + \sqrt{a_2^2 - 3a_1}}{3}$ and $\lambda_5 = \dfrac{-a_2 - \sqrt{a_2^2 - 3a_1}}{3}$, then $p(\lambda_4) < 0$ and $p(\lambda_5) > 0$.

Proof. If there exist three different $\lambda_i \in \mathbb{R}^+$, $(i = 1, 2, 3)$ such that $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 + (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda - \lambda_1\lambda_2\lambda_3$, then by equality of polynomials, we obtain:

- $a_2 = -(\lambda_1 + \lambda_2 + \lambda_3) < 0$,
- $a_0 = -\lambda_1\lambda_2\lambda_3 < 0$,
- $a_1 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 > 0$,

and the condition *1* hold.

Without loss of generality, we suppose that $0 < \lambda_1 < \lambda_2 < \lambda_3$[6], if we have a polynomial of degree 3 with three real, simple roots, then there exists two critical points, they are roots of the first derivative, i.e., the condition *2* holds. Now if the roots of the polynomial are positive then the critical points are positive too and the follow chain of inequalities hold $\lambda_1 < \lambda_5 < \lambda_2 < \lambda_4 < \lambda_3$. From the coefficient of λ^3 is 1, we have that $\lim_{\lambda\to\infty} p(\lambda) = \infty$ and $\lim_{\lambda\to-\infty} p(\lambda) = -\infty$, then for points less than λ_1 the polynomial is negative, we applied the Rolle theorem to the roots λ_1 and λ_2, therefore the polynomial has a maximum value between λ_1 and λ_2, and therefore $p(\lambda_5) > 0$. Similarly, for points greater than λ_3 the polynomial is positive, and we applied the Rolle theorem to the roots λ_2 and λ_3, and the polynomial has a minimum, this value is negative i.e. $p(\lambda_4) < 0$.

Conversely, we have that $p(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$ such that the conditions *1, 2* and *3* hold. The conditions *1,2* join to lemma 1 implies that $p'(\lambda)$ has two real, simple and positive roots given by :

$$\lambda_4 = \frac{-a_2 + \sqrt{a_2^2 - 3a_1}}{3}, \quad \lambda_5 = \frac{-a_2 - \sqrt{a_2^2 - 3a_1}}{3}. \quad (A.2)$$

[6] In this work, a chain of inequalities $a < b < c < \dots$, the first inequality is $a < b$, the second one is $b < c$ and so on.

First we observe that $0 < \lambda_5 < \lambda_4$, now computing $p''(\lambda_4) = 2\sqrt{a_2^2 - 3a_1} > 0$, this implies in λ_4 we have a minimum, whereas $p''(\lambda_5) = -2\sqrt{a_2^2 - 3a_1} < 0$ and then in λ_5 we have a maximum. We repeatedly applied the Intermediate Value theorem. From *1*, we have that $p(0) = a_0 < 0$ and from *3* it follows that $p(\lambda_5) > 0$, then we have a positive root. The condition *3*, $p(\lambda_4) < 0$ and $p(\lambda_5) > 0$, guarantee that exits a second root between λ_5 and λ_4, finally due to the coefficient to λ^3 is positive $p(\lambda)$ we have that $\lim_{\lambda\to\infty} p(\lambda) = \infty$ and this implies $p(\lambda_4) < 0$, then $p(\lambda)$ intersects the horizontal axis one more time in the third root. ■

Appendix B. Two-zero textures

In the above sections, we show that the matrices $M(B,F)$, $M(B,D)$ and $M(D,F)$ can have three positive, real and different eigenvalues, these matrices are diagonal by blocks (one block 1×1 and the other one 2×2). They can be diagonalized by matrices that are also diagonal by blocks.

Pay attention only in the 2×2 block. The mass matrix can be rewritten as:

$$M_{2\times2} = \begin{pmatrix} y & K \\ K & x \end{pmatrix}, \qquad (K = B, D, F). \tag{B.1}$$

This matrix has to be congruent with

$$\bar{M}_{2\times2} = \begin{pmatrix} m_i & 0 \\ 0 & m_j \end{pmatrix}, \qquad (i,j) = (1,2),(2,3),(1,3). \tag{B.2}$$

This implies the following relations among their elements:

$$x + y = m_i + m_j, \tag{B.3}$$
$$xy - K^2 = m_i m_j, \tag{B.4}$$

the solutions for x and y are:

$$x(K) = \mu_{ij} \pm \sqrt{\delta_{ij}^2 - K^2}, \tag{B.5}$$
$$y(K) = \mu_{ij} \mp \sqrt{\delta_{ij}^2 - K^2}, \tag{B.6}$$

where $\mu_{ij} = \dfrac{m_i + m_j}{2}$, if $m_i > m_j$, $\delta_{ij} = \dfrac{m_i - m_j}{2}$ and the parameter K has to satisfy $|K| \le \delta_{ij}$.

The matrix that diagonalize the matrix $M_{2\times2}$ always can be set as:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{B.7}$$

where: $\sin\theta = \dfrac{K}{\sqrt{K^2 + (y - m_i)^2}}$ and $\theta \in [0, \pi/4]$.

Appendix C. Perturbation Theory

In this appendix, we applied perturbation theory to texture formalism [7].

We start dividing the complete mass matrix in two parts

$$M = M_0 + \epsilon N, \tag{C.1}$$

where M_0 and N are known mass matrices and ϵ is a small parameter. We look for a O matrix that diagonalize the M matrix in the following way:

$$O^T M O = \bar{M}, \tag{C.2}$$

where $\bar{M}$ is a diagonal matrix.

We have three different versions of the perturbation method according to the way that O matrix is proposed, namely:

1. *Right Perturbation*, when the O matrix takes the form
$$O = O_0(1 + \epsilon X). \tag{C.3}$$
2. *Left Perturbation*, when the matrix O takes the form
$$O = (1 + \epsilon X)\, O_0. \tag{C.4}$$
3. *Left-Right Perturbation*, when the matrix O takes the form
$$O = (1 + \epsilon X)\, O_0\, (1 + \epsilon X). \tag{C.5}$$

Where the O_0 matrix diagonalizes the M_0 matrix ($O_0^T M_0 O_0 = \bar{M}$) and the X matrix is determined in this process.

From the orthogonality condition of the O matrix, it is found that X is an antisymmetric matrix $X^T = -X$ and $Y + Y^T = X^2$ for all cases.

Notation: We are considering $\bar{A} = O_0^T A\, O_0$ for any matrix A.

Appendix C.1. Right perturbation

Substituting the form the O matrix (Eq. C.3) into (Eq.C.2):

$$[O_0(1 + \epsilon X)]^T\ M\ [O_0(1 + \epsilon X)] = \bar{M}. \tag{C.6}$$

After some algebra, one gets, at first order in the ϵ parameter, that the X matrix has to satisfy:

$$\bar{N} = [X, \bar{M}]. \tag{C.7}$$

At second order in ϵ, the Y matrix has to satisfy:

$$\bar{N}X + X\bar{N} = [Y + Y^T, \bar{M}]. \tag{C.8}$$

Appendix C.2. Left perturbation

Substituting the form the O matrix (Eq. C.4) into (Eq.C.2):

$$[(1 + \epsilon X)O_0]^T\ M\ [(1 + \epsilon X)O_0] = \bar{M}. \tag{C.9}$$

After some algebra, one gets, at first order in the ϵ parameter, that the $\bar{X}$ matrix has to satisfy:

$$\bar{N} = [\bar{X}, \bar{M}]. \tag{C.10}$$

At second order in ϵ, the $\bar{Y}$ matrix has to satisfy:

$$\bar{N}\bar{X} + \bar{X}\bar{N} = [\bar{Y} + \bar{Y}^T, \bar{M}]. \tag{C.11}$$

[7] The first work in this direction was (Fritzsch et al. (2011)) and it applies non-hermitian perturbations to the 6-zero texture

Appendix C.3. Left-Right perturbation

Substituting the form the O matrix (Eq. C.5) into (Eq.C.2):

$$[(1+\epsilon X)O_0(1+\epsilon X)]^T \ M \ [(1+\epsilon X)O_0(1+\epsilon X)] = \bar{M}. \quad \text{(C.12)}$$

After some algebra, one gets, at first order in the ϵ parameter, that the $\bar{X}$ matrix has to satisfy:

$$\bar{N} = [X + \bar{X}, \bar{M}]. \quad \text{(C.13)}$$

References

Branco, G. C., Emmanuel-Costa, D., and Gonzalez Felipe, R. (2000). Texture zeros and weak basis transformations. *Phys. Lett. B*, 477:147–155.

Friedberg, S., Insel, A., and Spence, L. (2006). *Linear Algebra*. Pretice Hall, four edition edition.

Fritzsch, H. (1977a). Calculating the cabibbo angle. *Phys.Lett.*, 70:317.

Fritzsch, H. (1977b). Calculating the Cabibbo Angle. *Phys. Lett. B*, 70:436–440.

Fritzsch, H. and Xing, Z.-Z. (1998). The Mixing of quarks and leptons and neutrino degeneracy.

Fritzsch, H. and zhong Xing, Z. (2000). Mass and flavor mixing schemes of quarks and leptons. *Prog.Part.Nucl.Phys.*, 45:1–81.

Fritzsch, H. and zhong Xing, Z. (2003). Four zero texture of Hermitian quark mass matrices and current experimental tests. *Phys.Lett.*, B555:63–70.

Fritzsch, H., zhong Xing, Z., and Zhou, Y.-L. (2011). Non-hermitian perturbations to the fritzsch textures of lepton and quark mass matrices. *Physics Letters B*, 697(4):357–363.

Haussling, R. and Scheck, F. (1998). Triangular mass matrices of quarks and Cabibbo-Kobayashi-Maskawa mixing. *Phys. Rev. D*, 57:6656–6662.

Kobayashi, M. and Maskawa, T. (1973). CP Violation in the Renormalizable Theory of Weak Interaction. *Prog. Theor. Phys.*, 49:652–657.

Kuo, T.-K., Mansour, S. W., and Wu, G.-H. (1999). Classification and application of triangular quark mass matrices. *Phys. Lett. B*, 467:116–125.

Olive, K. et al. (2014). Review of Particle Physics. *Chin.Phys.*, C38:090001.

Weinberg, S. (1967). A Model of Leptons. *Phys. Rev. Lett.*, 19:1264–1266.