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A note on the problem of geodesics curves with fractional derivative of Atangana–Baleanu
Una nota sobre el problema de las curvas geodésicas con derivada fraccionaria de Atangana–Baleanu

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Resumen

Este artículo presenta algunas propiedades y relaciones que existen entre los operadores fraccionarios en el sentido de Riemann–Liouville y de Atangana–Baleanu. En particular, se presenta una demostración de la fórmula de integración por partes cuando la derivada fraccionaria de Atangana–Baleanu es considerada. Como una aplicación de estas propiedades, se analiza el problema clásico de la determinación de las curvas geodésicas en el plano considerando la derivada fraccionaria de Atangana–Baleanu. La introducción de la derivada fraccionaria en el funcional que describe el problema de optimización se realiza mediante el método de fraccionalización. Los resultados obtenidos se comparan con el problema clásico.

Palabras Clave: Operadores fraccionarios, integración por partes, cálculo de variaciones, curvas geodésicas.

Abstract

This paper presents some properties and relations that exist between fractional operators in the sense of Riemann–Liouville and Atangana–Baleanu. In particular, a proof of the integration by parts formula is presented when the fractional derivative of Atangana–Baleanu is considered. As an application of these properties, the classical problem of determining geodesic curves in the plane is analyzed considering the fractional derivative of Atangana–Baleanu. The introduction of the fractional derivative in the functional describing the optimization problem is performed by means of the fractionalization method. The obtained results are compared with the classical problem.

Keywords: Fractional operators, integration by parts, calculus of variations, geodesic curves.

1. Introduction

A fractional derivative D^α of order $\alpha > 0$ is an operator that generalizes the ordinary derivative in classical calculus. In this sense, the origin of fractional calculus arose as a problem of generalizing the notion of derivative $D^n x(t) = \frac{d^n}{dt^n} x(t)$ when n is a fraction; see Oldham and Spanier (1974). The possibility of generalizing the concept of derivative, as well as the concept of integral, is a problem that has been addressed by different mathematicians, among whom we can mention Euler, Laplace, Fourier, Riemann, Liouville, Abel, etc., and therefore, a fractional derivative has different definitions that generally do not coincide. A classification of some fractional derivatives can be consulted in Baleanu and Fernandez (2019) and Sales-Teodoro et al. (2019).

The diversity of such definitions is due to the fact that, in general, fractional operators have different kernel representations for different function spaces. This is the reason for using different methods to define a derivative, as well as an integral, of fractional order. Among all these methods to calculate a fractional integral, the most used approach is that of the Riemann–Liouville fractional integral, which is a generalization of the Cauchy formula for repeated integration of classical calculus; see Samko et al. (1993). However, in recent years alternative approaches for derivatives and integrals of fractional order have been proposed. One of the reasons for these new approaches is due to avoiding the singularity that is present at the end point of the integration interval for the Riemann–Liouville integral. In

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this sense, Atangana and Baleanu suggested replacing the kernel of the Riemann–Liouville operator with a function called the one-parameter Mittag–Leffler function; see Atangana and Balenu (2016). This approach was also proposed in Caputo and Fabrizio (2015) for another type of fractional derivative.

It is known that one of the first applications of fractional calculus is due to the research of Niels Henrik Abel in his study of one of the classical problems of the calculus of variations: the generalization of the tautochrone problem, see e.g. Podlubny et al. (2017). However, a formal analysis of a problem of fractional calculus of variations in non-conservative systems is presented in Riewe (1996, 1997), and after these investigations, a series of papers were published in which different applications of the fractional calculus of variations are studied; see e.g. Agrawal (2002), Baleanu (2009), Almeida et al. (2012), Coronel-Escamilla et al. (2016) and Chatibi et al. (2019).

In this paper, we analyze some relations between fractional operators in the sense of Riemann–Liouville and Atangana–Baleanu based on the results of Baleanu and Fernandez (2019). Using these properties, we obtain the integration by parts formula for the fractional derivative of Atangana–Baleanu in order to obtain the Euler–Lagrange equation that allows solving the fundamental problem of the calculus of variations with the fractional derivative of Atangana–Baleanu. As an application, we analyze the classical problem on the determination of geodesic curves in the plane using the fractional derivative of Atangana–Baleanu. The results obtained are compared with those shown in Chatibi et al. (2019).

2. Some properties and relations of the fractional derivatives of Riemann–Liouville and Atangana–Baleanu

We present the preliminaries on the fractional derivatives of Riemann–Liouville and Atangana–Baleanu and their main properties that we will use throughout this paper.

Let $p \in \mathbb{R}$ with $1 \leq p < \infty$, $k \in \mathbb{N} \cup \{\infty\}$ and $a, b \in \mathbb{R}$ with $a < b$. Let $\Omega = (a, b)$. We denote by $C^k(\bar{\Omega})$ the space of functions whose k-derivative is continuous in $\bar{\Omega}$, by $C_0(\bar{\Omega})$ the space of continuous functions with compact support on $\bar{\Omega}$, and by $L_p(\Omega)$ the space of functions for which the p-th power of its absolute value is Lebesgue-integrable on Ω. For each function $x \in L_p(\Omega)$, we consider the norm defined in the usual way by:

$$\|x\|_{L_p(\Omega)} = \left(\int_a^b |x(t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}}.$$

The abbreviation ${}_aI_t$ is sometimes used to denote the integral operator

$${}_aI_t x(t) = \int_a^t x(s) \, \mathrm{d}s,$$

where the integral is understood in the Lebesgue sense.

We consider the Sobolev space

$$W^{1,p}(\Omega) = \Big\{ x \in L_p(\Omega) \ : \ \text{exists } y \in L_p(\Omega) \text{ such that}$$
$$\int_a^b x(t) \frac{\mathrm{d}\varphi}{\mathrm{d}t}(t) \, \mathrm{d}t = - \int_a^b y(t)\varphi(t) \, \mathrm{d}t \text{ for all } \varphi \in C_0^\infty(\bar{\Omega}) \Big\},$$

where $C_0^\infty(\bar{\Omega}) = C^\infty(\bar{\Omega}) \cap C_0(\bar{\Omega})$. In particular, we consider the following notation that is used: $H^1(\Omega) = W^{1,2}(\Omega)$. It is well known that $C^1(\bar{\Omega}) \subset H^1(\Omega)$; see e.g. Brezis (2011).

We also recall the following particular case of the well known Fubini's Theorem: let $f : \Omega \times \Omega \to \mathbb{R}$ be a measurable function, then

$$\int_a^b \int_a^s f(s, t) \, \mathrm{d}t \mathrm{d}s = \int_a^b \int_t^b f(s, t) \, \mathrm{d}s \mathrm{d}t, \tag{1}$$

assuming that one of these integrals is absolutely convergent; see e.g. Samko et al. (1993).

2.1. Definition and properties of Riemann–Liouville fractional operators

It is well known that in traditional calculus, the integral of a function $x \in C^1(\bar{\Omega})$ can be considered as the inverse operation of differentiation, that is, it holds

$$\frac{\mathrm{d}}{\mathrm{d}t} {}_aI_t x(t) = x(t).$$

According to this reasoning, a traditional approach in fractional calculus is to first obtain a fractional integral and then, in a next step, obtain the fractional derivative of a function. This method is usually used in fractional calculus to obtain, as a particular case, the integration and differentiation operators in the Riemann–Liouville sense; see e.g. Ross (1975). We briefly review this approach.

If the operator ${}_aI_t$ is applied n-times to $x \in C(\bar{\Omega})$, that is, if we consider the operator defined by ${}_aI_t^n x(t) = {}_aI_t \, {}_aI_t^{n-1} x(t)$ for $n \geq 2$, then we get

$${}_aI_t^n x(t) = \int_a^t \int_a^{t_{n-1}} \cdots \int_a^{t_1} x(t_0) \, \mathrm{d}t_0 \mathrm{d}t_1 \cdots \mathrm{d}t_{n-1}.$$

The use of Cauchy's repeated integration formula can be used to reduce the previous expression to a single integral

$${}_aI_t^n x(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} x(s) \, \mathrm{d}s.$$

A similar argument can be used to show that

$${}_tI_b^n x(t) = \frac{1}{(n-1)!} \int_t^b (s-t)^{n-1} x(s) \, \mathrm{d}s.$$

If we introduce the following notation $(n-1)! = \Gamma(n)$ into these expressions, where $\Gamma(z) = \int_0^\infty s^{z-1} e^{-s} \, \mathrm{d}s$ is the Gamma function, then we obtain the following generalizations of fractional order integrals.

Definition 1 (Miller and Ross, 1993)**.** The left-sided fractional Riemann–Liouville integral of $x \in L_1(\Omega)$ of fractional order $\alpha \in (0, 1)$ is defined as:

$${}^{\mathrm{RL}}_{a}I_t^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s) \, \mathrm{d}s.$$

Similarly, the right-sided fractional Riemann–Liouville integral of $x \in L_1(\Omega)$ of fractional order $\alpha \in (0, 1)$ is defined as:

$${}^{\mathrm{RL}}_{t}I_b^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} x(s) \, \mathrm{d}s.$$

We note that, since $\alpha \in (0,1)$ and $x \in L_1(\Omega)$, the integrals ${}^{\mathrm{RL}}_{a}I^{\alpha}_{t}x(t)$ and ${}^{\mathrm{RL}}_{t}I^{\alpha}_{b}x(t)$ given in the Definition 1 exist for almost all $t \in \Omega$.

Sometimes it is required to define ${}^{\mathrm{RL}}_{a}I^{\alpha}_{t}x(t)$ and ${}^{\mathrm{RL}}_{t}I^{\alpha}_{b}x(t)$ when $\alpha \to 0$, in that case, we consider the following definitions:

$$ {}^{\mathrm{RL}}_{a}I^{0}_{t}x(t) = x(t), \tag{2}$$

$$ {}^{\mathrm{RL}}_{t}I^{0}_{b}x(t) = x(t). \tag{3}$$

The motivation for these definitions is due to the following reasoning given in Atanacković et al. (2014). If we assume that $x \in C^1(\bar{\Omega})$, after integrating by parts, we obtain

$$ {}^{\mathrm{RL}}_{a}I^{\alpha}_{t}x(t) = \frac{1}{\Gamma(\alpha+1)}\left(x(a)(t-a)^{\alpha} + \int_a^t (t-s)^{\alpha}\frac{\mathrm{d}x}{\mathrm{d}s}(s)\,\mathrm{d}s\right),$$

so that,

$$ \lim_{\alpha\to 0} {}^{\mathrm{RL}}_{a}I^{\alpha}_{t}x(t) = x(a) + \int_a^t \frac{\mathrm{d}x}{\mathrm{d}s}(s)\,\mathrm{d}s = x(t).$$

A similar argument is used to motivate the definition of the identity ${}^{\mathrm{RL}}_{t}I^{0}_{b}x(t) = x(t)$.

We have assumed in Definition 1 that $x \in L_1(\Omega)$, however, we can assume more generally that $x \in L_p(\Omega)$ with $p > 1$. In this case the following result is obtained.

Lemma 1. *If $x \in L_p(\Omega)$ with $p > 1$, then ${}^{\mathrm{RL}}_{a}I^{\alpha}_{t}x \in L_p(\Omega)$.*

Proof. We choose $p > 1$ and $q > 1$ so that $\frac{1}{p} + \frac{1}{q} = 1$ and observe that

$$ \left|{}^{\mathrm{RL}}_{a}I^{\alpha}_{t}x(t)\right| = \left|\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}x(s)\,\mathrm{d}s\right| \le \frac{1}{\Gamma(\alpha)}\int_a^t |t-s|^{\alpha-1}\,|x(s)|\,\mathrm{d}s = \frac{1}{\Gamma(\alpha)}\int_a^t |t-s|^{\frac{\alpha-1}{q}}\,|t-s|^{\frac{\alpha-1}{p}}\,|x(s)|\,\mathrm{d}s.$$

Now, using Hölder's inequality, see e.g. Bartle (1995), we observe that

$$ \int_a^t |t-s|^{\frac{\alpha-1}{q}}\,|t-s|^{\frac{\alpha-1}{p}}\,|x(s)|\,\mathrm{d}s \le \left(\int_a^t \left(|t-s|^{\frac{\alpha-1}{q}}\right)^q \mathrm{d}s\right)^{\frac{1}{q}}\left(\int_a^t \left(|t-s|^{\frac{\alpha-1}{p}}\,|x(s)|\right)^p \mathrm{d}s\right)^{\frac{1}{p}} \le \frac{(b-a)^{\frac{\alpha}{q}}}{\alpha^{\frac{1}{q}}}\left(\int_a^t |t-s|^{\alpha-1}|x(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}},$$

where the last inequality is obtained by noting that $t-a \le b-a$ for all $t \in \Omega$. It follows that

$$ \left|{}^{\mathrm{RL}}_{a}I^{\alpha}_{t}x(t)\right|^p \le \frac{(b-a)^{\alpha(p-1)}}{\Gamma(\alpha)^p\alpha^{p-1}}\int_a^t |t-s|^{\alpha-1}|x(s)|^p\,\mathrm{d}s.$$

As a consequence of the previous inequality, we observe that

$$ \left\|{}^{\mathrm{RL}}_{a}I^{\alpha}_{t}x\right\|^p_{L_p(\Omega)} = \int_a^b \left|{}^{\mathrm{RL}}_{a}I^{\alpha}_{t}x(t)\right|^p\,\mathrm{d}t, \le \frac{(b-a)^{\alpha(p-1)}}{\Gamma(\alpha)^p\alpha^{p-1}}\int_a^b\int_a^t |t-s|^{\alpha-1}|x(s)|^p\,\mathrm{d}s\mathrm{d}t.$$

On the other hand, using the particular case of Fubini's Theorem given in (1), we obtain

$$ \int_a^b\int_a^t |t-s|^{\alpha-1}|x(s)|^p\,\mathrm{d}s\mathrm{d}t = \int_a^b\int_t^b |t-s|^{\alpha-1}|x(s)|^p\,\mathrm{d}t\mathrm{d}s = \left(\int_t^b |t-s|^{\alpha-1}\,\mathrm{d}t\right)\left(\int_a^b |x(s)|^p\,\mathrm{d}s\right) \le \frac{(b-a)^{\alpha}}{\alpha}\int_a^b |x(s)|^p\,\mathrm{d}s,$$

where the validity of the inequality $b-s \le b-a$ for all $s \in (\bar{\Omega})$ has been used. Finally, if we consider the above inequality and take the p-th root, we get

$$ \left\|{}^{\mathrm{RL}}_{a}I^{\alpha}_{t}x\right\|_{L_p(\Omega)} \le \frac{(b-a)^{\alpha}}{\alpha\Gamma(\alpha)}\left(\int_a^b |x(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}},$$

that is, we obtain the inequality:

$$ \left\|{}^{\mathrm{RL}}_{a}I^{\alpha}_{t}x\right\|_{L_p(\Omega)} \le \frac{(b-a)^{\alpha}}{\alpha\Gamma(\alpha)}\|x\|_{L_p(\Omega)}, \tag{4}$$

which shows the desired result. □

An alternative method of obtaining the inequality (4) is shown in Samko et al. (1993). A consequence of Lemma 1 and the properties of the spaces $L_p(\Omega)$ with $p > 1$, is that if $x \in L_1(\Omega)$, then ${}^{\mathrm{RL}}_{a}I^{\alpha}_{t}x \in L_1(\Omega)$; see Bartle (1995).

It is known that the left-sided and right-sided fractional Riemann–Liouville integral satisfy the following semigroup properties:

$$ {}^{\mathrm{RL}}_{a}I^{\alpha}_{t}\,{}^{\mathrm{RL}}_{a}I^{\beta}_{t}x(t) = {}^{\mathrm{RL}}_{a}I^{\beta}_{t}\,{}^{\mathrm{RL}}_{a}I^{\alpha}_{t}x(t) = {}^{\mathrm{RL}}_{a}I^{\alpha+\beta}_{t}x(t), \tag{5}$$

$$ {}^{\mathrm{RL}}_{t}I^{\alpha}_{b}\,{}^{\mathrm{RL}}_{t}I^{\beta}_{b}x(t) = {}^{\mathrm{RL}}_{t}I^{\beta}_{b}\,{}^{\mathrm{RL}}_{t}I^{\alpha}_{b}x(t) = {}^{\mathrm{RL}}_{t}I^{\alpha+\beta}_{b}x(t); \tag{6}$$

see Miller and Ross (1993). The proof of the first identity in (5) is obtained from the corresponding definitions. In fact, we observe that

$$ {}^{\mathrm{RL}}_{a}I^{\alpha}_{t}\,{}^{\mathrm{RL}}_{a}I^{\beta}_{t}x(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\frac{1}{\Gamma(\beta)}\int_a^s (s-\tau)^{\beta-1}x(\tau)\,\mathrm{d}\tau\mathrm{d}s = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^s (t-s)^{\alpha-1}(s-\tau)^{\beta-1}x(\tau)\,\mathrm{d}\tau\mathrm{d}s = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t x(\tau)\int_\tau^t (t-s)^{\alpha-1}(s-\tau)^{\beta-1}\,\mathrm{d}s\mathrm{d}\tau,$$

where we have used the particular case of Fubini's Theorem given in (1). If we choose $s = \sigma(t-\tau)+\tau$, then we obtain that

$$ \int_s^t (t-s)^{\alpha-1}(s-\tau)^{\beta-1}\,\mathrm{d}s = (t-\tau)^{\alpha+\beta-1}\int_0^1 (1-\sigma)^{\alpha-1}\sigma^{\beta-1}\,\mathrm{d}\sigma = (t-\tau)^{\alpha+\beta-1}B(\alpha,\beta),$$

where $B(\alpha,\beta) = \int_0^1 (1-\sigma)^{\alpha-1}\sigma^{\beta-1}\,\mathrm{d}\sigma$ is the Beta function. On the other hand, since $B(\alpha,\beta)\Gamma(\alpha+\beta) = \Gamma(\alpha)\Gamma(\beta)$, see e.g. Lebedev (1965), we finally obtain that

$$ {}^{\mathrm{RL}}_{a}I^{\alpha}_{t}\,{}^{\mathrm{RL}}_{a}I^{\beta}_{t}x(t) = \frac{1}{\Gamma(\alpha+\beta)}\int_a^t (t-\tau)^{\alpha+\beta-1}x(\tau)\,\mathrm{d}\tau = {}^{\mathrm{RL}}_{a}I^{\alpha+\beta}_{t}x(t).$$

A similar argument is used to show the second part of the identity (6) and the identities in (6). In particular, we have

$$ {}^{\mathrm{RL}}_{a}I^{\alpha k}_{t}x(t) = {}^{\mathrm{RL}}_{a}I^{\alpha}_{t}\,{}^{\mathrm{RL}}_{a}I^{\alpha(k-1)}_{t}x(t), \quad k \in \mathbb{N}. \tag{7} $$

The above relation shows that fractional Riemann–Liouville integrals have a nice composition property, namely, the fractional Riemann–Liouville integral of a fractional Riemann–Liouville integral is a fractional Riemann–Liouville integral of a certain corresponding order.

Now we can introduce the Riemann–Liouville fractional derivative.

Definition 2 (Miller and Ross, 1993)**.** The left-sided fractional Riemann–Liouville derivative of $x \in L_1(\Omega)$ is defined as:

$$ {}^{\mathrm{RL}}_{a}D^{\alpha}_{t}x(t) = \frac{1}{\Gamma(\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t (t-s)^{\alpha-1}x(s)\,\mathrm{d}s. $$

Analogously, the right-sided fractional Riemann–Liouville derivative of $x \in L_1(\Omega)$ is defined as:

$$ {}^{\mathrm{RL}}_{t}D^{\alpha}_{b}x(t) = \frac{1}{\Gamma(\alpha)}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^b (t-s)^{\alpha-1}x(s)\,\mathrm{d}s. $$

The following identity will be used later.

Lemma 2. *Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $x \in L_p(\Omega)$ and $y \in L_q(a, b)$, then*

$$ \int_a^b x(t)\,{}^{\mathrm{RL}}_{a}I^{\alpha}_{t}y(t)\,\mathrm{d}t = \int_a^b y(t)\,{}^{\mathrm{RL}}_{t}I^{\alpha}_{b}x(t)\,\mathrm{d}t. $$

Proof. We first observe that from Hölder's inequality, see e.g. Bartle (1995), it follows that $x\,{}^{\mathrm{RL}}_{a}I^{\alpha}_{t}y$ and $y\,{}^{\mathrm{RL}}_{a}I^{\alpha}_{t}x$ are elements of $L_1(\Omega)$. Thus, from the Definition 1 and the particular case of Fubini's Theorem given in (1), we obtain

$$ \begin{aligned} \int_a^b x(t)\,{}^{\mathrm{RL}}_{a}I^{\alpha}_{t}y(t)\,\mathrm{d}t &= \int_a^b\int_a^t \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}x(t)y(s)\,\mathrm{d}s\mathrm{d}t \\ &= \int_a^b\int_t^b \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}x(t)y(s)\,\mathrm{d}t\mathrm{d}s \\ &= \int_a^b y(s)\frac{1}{\Gamma(\alpha)}\int_t^b (t-s)^{\alpha-1}x(t)\,\mathrm{d}t\mathrm{d}s \\ &= \int_a^b y(t)\,{}^{\mathrm{RL}}_{t}I^{\alpha}_{b}x(t)\,\mathrm{d}t, \end{aligned} $$

which shows the desired result. □

2.2. Definition and properties of Atangana–Baleanu fractional operators

In this section we recall some basic notions related to the Riemann–Liouville and Atangana–Baleanu fractional derivatives. We start with the definition of the Atangana–Baleanu fractional derivative.

Definition 3 (Atangana and Balenu, 2016)**.** The left-sided Atangana–Baleanu fractional integral of $x \in H^1(\Omega)$ is defined by:

$$ {}^{\mathrm{AB}}_{a}I^{\alpha}_{t}x(t) = \frac{1-\alpha}{B(\alpha)}x(t) + \frac{\alpha}{B(\alpha)}\,{}^{\mathrm{RL}}_{a}I^{\alpha}_{t}x(t), $$

where $B(x)$ is a function that satisfies: $B(0) = B(1) = 1$. Analogously, the right-sided fractional Atangana–Baleanu integral of $x \in H^1(\Omega)$ is definided by:

$$ {}^{\mathrm{AB}}_{t}I^{\alpha}_{b}x(t) = \frac{1-\alpha}{B(\alpha)}x(t) + \frac{\alpha}{B(\alpha)}\,{}^{\mathrm{RL}}_{t}I^{\alpha}_{b}x(t). $$

We observe that if $x \in H^1(\Omega)$, then

$$ \begin{aligned} {}^{\mathrm{AB}}_{a}I^{0}_{t}x(t) &= \lim_{\alpha\to 0}{}^{\mathrm{AB}}_{a}I^{\alpha}_{t}x(t) \\ &= \lim_{\alpha\to 0}\frac{1-\alpha}{B(\alpha)}x(t) + \lim_{\alpha\to 0}\frac{\alpha}{B(\alpha)}\,{}^{\mathrm{RL}}_{t}I^{\alpha}_{b}x(t) \\ &= x(t), \end{aligned} $$

since $B(0) = 1$ and ${}^{\mathrm{RL}}_{a}I^{0}_{t}x(t) = x(t)$, according to the definition given in (2). In a completely analogous way we have that: ${}^{\mathrm{AB}}_{t}I^{0}_{b}x(t) = x(t)$.

The following properties of the Atangana–Baleanu fractional integral are valid.

Lemma 3. *For each $x \in H^1(\Omega)$ it holds*

$$ {}^{\mathrm{AB}}_{a}I^{\alpha}_{t}\,{}^{\mathrm{RL}}_{a}I^{\beta}_{t}x(t) = {}^{\mathrm{RL}}_{a}I^{\beta}_{t}\,{}^{\mathrm{AB}}_{a}I^{\alpha}_{t}x(t), \tag{8} $$

$$ {}^{\mathrm{AB}}_{t}I^{\alpha}_{b}\,{}^{\mathrm{RL}}_{t}I^{\beta}_{b}x(t) = {}^{\mathrm{RL}}_{t}I^{\beta}_{b}\,{}^{\mathrm{AB}}_{t}I^{\alpha}_{b}x(t). \tag{9} $$

Proof. From Definition 3 and the identity (5), we obtain:

$$ \begin{aligned} {}^{\mathrm{AB}}_{a}I^{\alpha}_{t}\,{}^{\mathrm{RL}}_{a}I^{\beta}_{t}x(t) &= \frac{1-\alpha}{B(\alpha)}\,{}^{\mathrm{RL}}_{a}I^{\beta}_{t}x(t) + \frac{\alpha}{B(\alpha)}\,{}^{\mathrm{RL}}_{a}I^{\alpha}_{t}\,{}^{\mathrm{RL}}_{a}I^{\beta}_{t}x(t) \\ &= \frac{1-\alpha}{B(\alpha)}\,{}^{\mathrm{RL}}_{a}I^{\beta}_{t}x(t) + \frac{\alpha}{B(\alpha)}\,{}^{\mathrm{RL}}_{a}I^{\beta}_{t}\,{}^{\mathrm{RL}}_{a}I^{\alpha}_{t}x(t) \\ &= {}^{\mathrm{RL}}_{a}I^{\beta}_{t}\left(\frac{1-\alpha}{B(\alpha)}x(t) + \frac{\alpha}{B(\alpha)}\,{}^{\mathrm{RL}}_{a}I^{\alpha}_{t}x(t)\right) \\ &= {}^{\mathrm{RL}}_{a}I^{\beta}_{t}\,{}^{\mathrm{AB}}_{a}I^{\alpha}_{t}x(t). \end{aligned} $$

A similar argument is used to show the identity in (9). □

The following result shows that the semigroup property is also valid for the Atangana–Baleanu fractional integral.

Lemma 4. *For each $x \in H^1(\Omega)$ it holds*

$$ {}^{\mathrm{AB}}_{a}I^{\alpha}_{t}\,{}^{\mathrm{AB}}_{a}I^{\beta}_{t}x(t) = {}^{\mathrm{AB}}_{a}I^{\beta}_{t}\,{}^{\mathrm{AB}}_{a}I^{\alpha}_{t}x(t), \tag{10} $$

$$ {}^{\mathrm{AB}}_{t}I^{\alpha}_{b}\,{}^{\mathrm{AB}}_{t}I^{\beta}_{b}x(t) = {}^{\mathrm{AB}}_{t}I^{\beta}_{b}\,{}^{\mathrm{AB}}_{t}I^{\alpha}_{b}x(t). \tag{11} $$

Proof. From the Definition 4 and Lemma 3, we obtain:

$$ \begin{aligned} {}^{\mathrm{AB}}_{a}I^{\alpha}_{t}\,{}^{\mathrm{AB}}_{a}I^{\beta}_{t}x(t) &= \frac{1-\alpha}{B(\alpha)}\,{}^{\mathrm{AB}}_{a}I^{\beta}_{t}x(t) + \frac{\alpha}{B(\alpha)}\,{}^{\mathrm{RL}}_{a}I^{\alpha}_{t}\,{}^{\mathrm{AB}}_{a}I^{\beta}_{t}x(t). \\ &= \frac{1-\alpha}{B(\alpha)}\,{}^{\mathrm{AB}}_{a}I^{\beta}_{t}x(t) + \frac{\alpha}{B(\alpha)}\,{}^{\mathrm{AB}}_{a}I^{\beta}_{t}\,{}^{\mathrm{RL}}_{a}I^{\alpha}_{t}x(t) \\ &= {}^{\mathrm{AB}}_{a}I^{\beta}_{t}\left[\frac{1-\alpha}{B(\alpha)}x(t) + \frac{\alpha}{B(\alpha)}\,{}^{\mathrm{RL}}_{a}I^{\alpha}_{t}x(t)\right] \\ &= {}^{\mathrm{AB}}_{a}I^{\beta}_{t}\,{}^{\mathrm{AB}}_{a}I^{\alpha}_{t}x(t). \end{aligned} $$

A similar argument is used to show the identity in (11). □

Lemma 5. *For each $x, y \in H^1(\Omega)$ it holds*

$$ \int_a^b x(t)\,{}^{\mathrm{AB}}_{a}I^{\alpha}_{t}y(t)\,\mathrm{d}t = \int_a^b y(t)\,{}^{\mathrm{AB}}_{t}I^{\alpha}_{b}x(t)\,\mathrm{d}t. $$

Proof. Applying the Definition 4 and the Lemma 2, we obtain the following identities

$$\begin{aligned}\int_a^b x(t)\,{}^{\mathrm{AB}}_{\ a}I^{\alpha}_{t}y(t)\,\mathrm{d}t &= \int_a^b x(t)\left(\frac{1-\alpha}{B(\alpha)}y(t)+\frac{\alpha}{B(\alpha)}\,{}^{\mathrm{RL}}_{\ a}I^{\alpha}_{t}y(t)\right)\mathrm{d}t\\ &= \frac{1-\alpha}{B(\alpha)}\int_a^b x(t)y(t)\,\mathrm{d}t+\frac{\alpha}{B(\alpha)}\int_a^b x(t)\,{}^{\mathrm{RL}}_{\ a}I^{\alpha}_{t}y(t)\mathrm{d}t\\ &= \frac{1-\alpha}{B(\alpha)}\int_a^b x(t)y(t)\mathrm{d}t+\frac{\alpha}{B(\alpha)}\int_a^b y(t)\,{}^{\mathrm{RL}}_{\ t}I^{\alpha}_{b}x(t)\mathrm{d}t\\ &= \int_a^b y(t)\left(\frac{1-\alpha}{B(\alpha)}x(t)+\frac{\alpha}{B(\alpha)}\,{}^{\mathrm{RL}}_{\ t}I^{\alpha}_{b}x(t)\right)\mathrm{d}t\\ &= \int_a^b y(t)\,{}^{\mathrm{AB}}_{\ t}I^{\alpha}_{b}x(t)\,\mathrm{d}t,\end{aligned}$$

which shows the desired result. □

Next we introduce one of the most important functions of fractional calculus.

Definition 4 (Gorenflo et al., 2020)**.** The Mittag–Leffler function of a parameter $\alpha \in (0,1)$ is defined as:

$$E_\alpha(z)=\sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(\alpha k+1)},\quad z\in\mathbb{C}.$$

We observe that the series that defines the Mittag–Leffler function $E_\alpha(z)$ converge on the whole complex plane, and such a function is a natural generalization of the exponential function with base e, that is, $e^z = E_1(z)$.

An application of the Mittag–Leffler function is that it forms the basis of the definition of the Atangana–Baleanu fractional derivative, which is introduced as follows.

Definition 5 (Atangana and Balenu, 2016)**.** The left-sided fractional Atangana–Baleanu derivative in the Riemann–Liouville sense of a function $x \in H^1(\Omega)$ is defined as:

$${}^{\mathrm{ABR}}_{\ \ a}D^{\alpha}_{t}x(t)=\frac{B(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t x(s)E_\alpha\left(-\mu_\alpha(t-s)^\alpha\right)\,\mathrm{d}s,$$

where $\mu_\alpha = \frac{\alpha}{1-\alpha}$ and $B(x)$ is a function that satisfies the identity: $B(0) = B(1) = 1$. Analogously, the right-sided fractional Atangana–Baleanu derivative in the Riemann–Liouville sense is defined as:

$${}^{\mathrm{ABR}}_{\ \ t}D^{\alpha}_{b}x(t)=-\frac{B(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^b x(s)E_\alpha\left(-\mu_\alpha(s-t)^\alpha\right)\,\mathrm{d}s.$$

There are definitions of the left- and right-sided Atangana–Baleanu fractional derivatives in the sense of Caputo that are not used here, but which are closely related to the fractional derivatives introduced in Definition 5, see Atangana and Balenu (2016). On the other hand, it is sometimes appropriate to have alternate representations of the left- and right-sided Atangana–Baleanu fractional derivatives other than those given in Definition 5. These representations are obtained by considering the Mittag–Leffler function as a power series. Indeed, from Definition 4, we alternatively obtain the following identity for the left-sided fractional Atangana–Baleanu derivative in the Riemann–Liouville sense; see Baleanu and Fernandez (2018):

$$\begin{aligned}{}^{\mathrm{ABR}}_{\ \ a}D^{\alpha}_{t}x(t) &= \frac{B(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t x(s)E_\alpha\left(-\mu_\alpha(t-s)^\alpha\right)\,\mathrm{d}s\\ &= \frac{B(\alpha)}{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t x(s)\sum_{k=0}^{\infty}\frac{(-\mu_\alpha)^k(t-s)^{\alpha k}}{\Gamma(\alpha k+1)}\,\mathrm{d}s\\ &= \frac{B(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\frac{(-\mu_\alpha)^k}{\Gamma(\alpha k+1)}\frac{\mathrm{d}}{\mathrm{d}t}\int_a^t x(s)(t-s)^{\alpha k}\,\mathrm{d}s\\ &= \frac{B(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\frac{(-\mu_\alpha)^k}{\Gamma(\alpha k+1)}\int_a^t x(s)\frac{\mathrm{d}}{\mathrm{d}t}(t-s)^{\alpha k}\,\mathrm{d}s\\ &= \frac{B(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}\frac{(-\mu_\alpha)^k}{\Gamma(\alpha k)}\int_a^t x(s)(t-s)^{\alpha k-1}\,\mathrm{d}s\\ &= \frac{B(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(-\mu_\alpha)^k\,{}^{\mathrm{RL}}_{\ a}I^{\alpha k}_{t}x(t),\end{aligned}$$

that is, we obtain the following alternative representation for the left-sided fractional Atangana–Baleanu derivative in terms of the left-sided fractional Riemann–Liouville integral:

$${}^{\mathrm{ABR}}_{\ \ a}D^{\alpha}_{t}x(t)=\frac{B(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(-\mu_\alpha)^k\,{}^{\mathrm{RL}}_{\ a}I^{\alpha k}_{t}x(t). \tag{12}$$

A similar expression for the right-sided fractional Atangana–Baleanu derivative in terms of the right-sided fractional Riemann–Liouville integral is obtained:

$${}^{\mathrm{ABR}}_{\ \ t}D^{\alpha}_{b}x(t)=\frac{B(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(-\mu_\alpha)^k\,{}^{\mathrm{RL}}_{\ t}I^{\alpha k}_{b}x(t). \tag{13}$$

We observe that the terms of the series (12) and (13) are well-defined for almost all $t \in \Omega$, since the fractional integrals in the Rieman–Liouville sense ${}^{\mathrm{RL}}_{\ a}I^{\alpha k}_{t}x(t)$ and ${}^{\mathrm{RL}}_{\ t}I^{\alpha k}_{b}x(t)$ are well-defined for almost all $t \in \Omega$ and $k \in \mathbb{N}$, which follows from (7). Furthermore, it is clear that if ${}^{\mathrm{ABR}}_{\ \ a}D^{\alpha}_{t}x(t)$ and ${}^{\mathrm{ABR}}_{\ \ t}D^{\alpha}_{b}x(t)$ exist for almost all $t \in \Omega$, then the series on the right-hand sides of (12) and (13) are convergent for almost all $t \in \Omega$.

The following result shows the relation between the fractional integral and the fractional derivative in the Atangana–Baleanu sense.

Lemma 6. *Let $x \in H^1(\Omega)$ and $\alpha \in (0,1)$, then it holds*

$${}^{\mathrm{ABR}}_{\ \ a}D^{\alpha}_{t}\,{}^{\mathrm{AB}}_{\ a}I^{\alpha}_{t}x(t)={}^{\mathrm{AB}}_{\ a}I^{\alpha}_{t}\,{}^{\mathrm{ABR}}_{\ \ a}D^{\alpha}_{t}x(t), \tag{14}$$

$${}^{\mathrm{ABR}}_{\ \ t}D^{\alpha}_{b}\,{}^{\mathrm{AB}}_{\ t}I^{\alpha}_{b}x(t)={}^{\mathrm{AB}}_{\ t}I^{\alpha}_{b}\,{}^{\mathrm{ABR}}_{\ \ t}D^{\alpha}_{b}x(t). \tag{15}$$

Proof. In the first part, we apply the identity (12) and Lemma 3

$$\begin{aligned}{}^{\mathrm{ABR}}_{\ \ a}D^{\alpha}_{t}\,{}^{\mathrm{AB}}_{\ a}I^{\alpha}_{t}x(t) &= \frac{B(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(-\mu_\alpha)^k\,{}^{\mathrm{RL}}_{\ a}I^{\alpha k}_{t}\,{}^{\mathrm{AB}}_{\ a}I^{\alpha}_{t}x(t)\\ &= \frac{B(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(-\mu_\alpha)^k\,{}^{\mathrm{AB}}_{\ a}I^{\alpha}_{t}\,{}^{\mathrm{RL}}_{\ a}I^{\alpha k}_{t}x(t)\\ &= {}^{\mathrm{AB}}_{\ a}I^{\alpha}_{t}\left(\frac{B(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(-\mu_\alpha)^k\,{}^{\mathrm{RL}}_{\ a}I^{\alpha k}_{t}x(t)\right)\\ &= {}^{\mathrm{AB}}_{\ a}I^{\alpha}_{t}\,{}^{\mathrm{ABR}}_{\ \ a}D^{\alpha}_{t}x(t).\end{aligned}$$

The identity in (17) is obtained analogously. □

An extension of the identities (14) and (15) that is obtained from (12) and (13) is the following: if ${}^{\text{ABR}}_{\ a}D^{\alpha}_{t}x(t)$ and ${}^{\text{ABR}}_{\ t}D^{\alpha}_{b}x(t)$ exist for almost all $t \in \Omega$, then

$$ {}^{\text{ABR}}_{\ a}D^{\alpha}_{t}\,{}^{\text{AB}}_{\ a}I^{\alpha}_{t}x(t) = {}^{\text{AB}}_{\ a}I^{\alpha}_{t}\,{}^{\text{ABR}}_{\ a}D^{\alpha}_{t}x(t) = x(t), \tag{16} $$

$$ {}^{\text{ABR}}_{\ t}D^{\alpha}_{b}\,{}^{\text{AB}}_{\ t}I^{\alpha}_{b}x(t) = {}^{\text{AB}}_{\ t}I^{\alpha}_{b}\,{}^{\text{ABR}}_{\ t}D^{\alpha}_{b}x(t) = x(t). \tag{17} $$

In reality, to show the identity (16) it is enough to observe that for almost all $t \in \Omega$:

$$ \begin{aligned} &{}^{\text{ABR}}_{\ a}D^{\alpha}_{t}\,{}^{\text{AB}}_{\ a}I^{\alpha}_{t}x(t) \\ &= {}^{\text{ABR}}_{\ a}D^{\alpha}_{t}\left(\frac{1-\alpha}{B(\alpha)}x(t) + \frac{\alpha}{B(\alpha)}\,{}^{\text{RL}}_{\ a}I^{\alpha}_{t}x(t)\right) \\ &= \frac{1-\alpha}{B(\alpha)}\,{}^{\text{ABR}}_{\ a}D^{\alpha}_{t}x(t) + \frac{\alpha}{B(\alpha)}\,{}^{\text{ABR}}_{\ a}D^{\alpha}_{t}\,{}^{\text{RL}}_{\ a}I^{\alpha}_{t}x(t) \\ &= \sum_{k=0}^{\infty}(-\mu_\alpha)^k\,{}^{\text{RL}}_{\ a}I^{\alpha k}_{t}x(t) - \sum_{k=0}^{\infty}(-\mu_\alpha)^{k+1}\,{}^{\text{RL}}_{\ a}I^{\alpha(k+1)}_{t}x(t) \\ &= {}^{\text{RL}}_{\ a}I^{0}_{t}x(t) + \sum_{k=0}^{\infty}(-\mu_\alpha)^{k+1}\,{}^{\text{RL}}_{\ a}I^{\alpha(k+1)}_{t}x(t) \\ &\qquad - \sum_{k=0}^{\infty}(-\mu_\alpha)^{k+1}\,{}^{\text{RL}}_{\ a}I^{\alpha(k+1)}_{t}x(t) \\ &= x(t), \end{aligned} $$

where we have used the identity (2) and the fact that the series $\sum_{k=0}^{\infty}(-\mu_\alpha)^{k+1}\,{}^{\text{RL}}_{\ a}I^{\alpha(k+1)}_{t}x(t)$ is convergent for almost all $t \in \Omega$ since ${}^{\text{ABR}}_{\ a}D^{\alpha}_{t}x(t)$ exists for almos all $t \in \Omega$. The identity (17) is shown analogously.

We conclude this section with the following result that will be important in Sect. 3: the integration by parts formula.

Lemma 7. *For each $x, y \in H^1(\Omega)$ it holds*

$$ \int_a^b x(t)\,{}^{\text{ABR}}_{\ a}D^{\alpha}_{t}y(t)\,\mathrm{d}t = \int_a^b y(t)\,{}^{\text{ABR}}_{\ t}D^{\alpha}_{b}x(t)\,\mathrm{d}t. $$

Proof. By the identity (12) and the particular case of Fubini's Theorem given in (1), we obtain

$$ \begin{aligned} &\int_a^b x(t)\,{}^{\text{ABR}}_{\ a}D^{\alpha}_{t}y(t)\,\mathrm{d}t \\ &= \int_a^b x(t)\left(\frac{B(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(-\mu_\alpha)^k\,{}^{\text{RL}}_{\ a}I^{\alpha k}_{t}y(t)\right)\mathrm{d}t \\ &= \frac{B(\alpha)}{1-\alpha}\int_a^b x(t)\left(\sum_{k=0}^{\infty}\frac{(-\mu_\alpha)^k}{\Gamma(\alpha k)}\int_a^t (t-s)^{\alpha k-1}y(s)\,\mathrm{d}s\right)\mathrm{d}t \\ &= \frac{B(\alpha)}{1-\alpha}\int_a^b\int_a^t\sum_{k=0}^{\infty}\frac{(-\mu_\alpha)^k}{\Gamma(\alpha k)}(t-s)^{\alpha k-1}x(t)y(s)\,\mathrm{d}s\mathrm{d}t \\ &= \frac{B(\alpha)}{1-\alpha}\int_a^b\int_s^b\sum_{k=0}^{\infty}\frac{(-\mu_\alpha)^k}{\Gamma(\alpha k)}(t-s)^{\alpha k-1}x(t)y(s)\,\mathrm{d}t\mathrm{d}s \\ &= \frac{B(\alpha)}{1-\alpha}\int_a^b y(s)\left(\sum_{k=0}^{\infty}\frac{(-\mu_\alpha)^k}{\Gamma(\alpha k)}\int_s^b (t-s)^{\alpha k-1}x(t)\,\mathrm{d}t\right)\mathrm{d}s \\ &= \int_a^b y(s)\left(\frac{B(\alpha)}{1-\alpha}\sum_{k=0}^{\infty}(-\mu_\alpha)^k\,{}^{\text{RL}}_{\ s}I^{\alpha k}_{b}x(s)\right)\mathrm{d}s \\ &= \int_a^b y(s)\,{}^{\text{ABR}}_{\ s}D^{\alpha}_{b}x(s)\,\mathrm{d}s, \end{aligned} $$

which shows the result. □

There are other versions of the integration by parts formula that are known in the literature, among which we can mention: Abdeljawad and Baleanu (2017), Abdeljawad et al. (2019) and Chatibi et al. (2019).

3. The fundamental problem of the calculus of variations with fractional derivatives

It is known that the calculus of variations is a generalization of elementary calculus in which the main problem consists of finding the maximum or minimum values of continuous functionals that are defined in some pre-established space of functions; see e.g. Freguglia and Giaquinta (2016) and Troutman (1996). A main feature of the calculus of variations is that some phenomena can be modeled by a functional $J\colon D(\Omega) \to \mathbb{R}$ which is described by an integral operator of the form

$$ J[x] = \int_a^b L(t, x(t), \dot{x}(t))\,\mathrm{d}t, $$

where $D(\Omega)$ is some set of functions defined *a priori* and $\Omega = (a, b)$ is an interval such that $a < b$ (the point b may be known or unknown). The scalar function $L\colon \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is called a Lagrange function, and may not depend on each of its arguments. In this sense, the fundamental problem of classical calculus of variations consists in finding the extreme values, if they exist, of the functional $J[x]$ on the set $D(\Omega)$, that is, determining a function $\bar{x}(t) \in D(\Omega)$ such that

$$ J[\bar{x}] = \inf_{x(t)\in D(\Omega)} J[x]. $$

(Equivalently, we can consider the problem of finding a function $\bar{\bar{x}} \in D(\Omega)$ such that $J[\bar{\bar{x}}] = \sup_{x(t)\in D(\Omega)} J[x]$.) It is well known that one of the tools that allows solving this optimization problem is formulated in the following result, known as Lagrange's Lemma.

Lemma 8 (Troutman, 1996)**.** *If $g \in C(\bar{\Omega})$ and*

$$ \int_a^b g(t)h(t)\,\mathrm{d}t = 0, $$

for all $h \in C(\bar{\Omega})$ such that $h(a) = h(b) = 0$, then $g(t) \equiv 0$ on $\bar{\Omega}$.

In this paper, we consider a generalization of the fundamental problem of the calculus of variations by considering a functional $J\colon D(\Omega) \to \mathbb{R}$ defined by

$$ J[x] = \int_a^b L(t, x(t), {}^{\text{ABR}}_{\ a}D^{\alpha}_{t}x(t))\,\mathrm{d}t, \tag{18} $$

where

$$ D(\Omega) = \{x \in C^1(\bar{\Omega}) \,:\, x(a) = x_a \text{ and } x(b) = x_b\}, $$

and where $x_a, x_b \in \mathbb{R}$ are constant values. In this case, the fundamental problem of the calculus of variations consists of solving the following optimization problem:

$$ \inf_{x(t)\in D(\Omega)} J[x]. \tag{19} $$

It is well known that if $\bar{x}(t) \in D(\Omega)$ allows solving this optimization problem, then the following necessary optimality condition is satisfied:

$$\lim_{\epsilon\to 0}\frac{J[\bar{x}+\epsilon h]-J[\bar{x}]}{\epsilon}=0,$$

where $h\colon \Omega \to \mathbb{R}$ is a function such that $\bar{x}(t)+h(t)\in D(\Omega)$, and as a consequence, such a function satisfies the conditions $h(a)=h(b)=0$; see e.g. Troutman (1996). If this condition is applied to the functional defined in (18), and Lemma 8 is applied, then the following chain of equalities is obtained:

$$\begin{aligned}
\lim_{\epsilon\to 0}\frac{J[\bar{x}+\epsilon h]-J[\bar{x}]}{\epsilon} &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}\epsilon} L(t,\bar{x}(t)+\epsilon h(t), {}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t(\bar{x}(t)+\epsilon h(t)))\Big|_{\epsilon=0}\,\mathrm{d}t\\
&= \int_a^b \left(\frac{\partial L}{\partial x}(t,\bar{x}(t),{}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t\bar{x}(t))h(t)+\frac{\partial L}{\partial\, {}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t x}(t,\bar{x}(t),{}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t\bar{x}(t))\,{}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t h(t)\right)\mathrm{d}t\\
&= \int_a^b \frac{\partial L}{\partial x}(t,\bar{x}(t),{}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t\bar{x}(t))h(t)\,\mathrm{d}t+\int_a^b \frac{\partial L}{\partial\, {}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t x}(t,\bar{x}(t),{}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t\bar{x}(t))\,{}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t h(t)\,\mathrm{d}t\\
&= \int_a^b \frac{\partial L}{\partial x}(t,\bar{x}(t),{}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t\bar{x}(t))h(t)\,\mathrm{d}t+\int_a^b {}^{\mathrm{ABR}}_{\ \ t}D^\alpha_b\frac{\partial L}{\partial\, {}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t x}(t,\bar{x}(t),{}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t\bar{x}(t))h(t)\,\mathrm{d}t\\
&= \int_a^b \left(\frac{\partial L}{\partial x}(t,\bar{x}(t),{}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t\bar{x}(t))+{}^{\mathrm{ABR}}_{\ \ t}D^\alpha_b\frac{\partial L}{\partial\, {}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t x}(t,\bar{x}(t),{}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t\bar{x}(t))\right)h(t)\,\mathrm{d}t=0.
\end{aligned}$$

Now, according to Lemma 8, if $\bar{x}\in D(\Omega)$ solves the optimization problem (19), then the following equation, called Euler–Lagrange, is obtained:

$$\frac{\partial L}{\partial x}\left(t,\bar{x},{}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t\bar{x}\right)+{}^{\mathrm{ABR}}_{\ \ t}D^\alpha_b\frac{\partial L}{\partial\, {}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t x}\left(t,\bar{x},{}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t\bar{x}\right)=0. \quad (20)$$

Finally, we consider a classic example of calculus of variations: the problem of determining geodesic curves in the plane. To analyze this example, we use the fractionalization method discussed in Gómez-Aguilar et al. (2014) to introduce fractional derivatives into the functional that models the geodesic problem. This method consists of introducing a new parameter σ that represents the components of the time in the fractional Atanga–Baleanu derivative with a suitable dimensionality, that is, we assume the following assignment:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mapsto\frac{1}{\sigma^{1-\alpha}}\,{}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t,\quad \alpha\in(0,1).$$

Example 1. We consider the following functional from the geodesic problem defined by

$$J[x]=\int_a^b L(t,x(t),\dot{x}(t))\,\mathrm{d}t=\int_a^b\sqrt{1+\dot{x}^2(t)}\,\mathrm{d}t,$$

with $x\colon\Omega\to\mathbb{R}$ a differentiable function such that $x(a)=x_a$ and $x(b)=x_b$; see e.g. Troutman (1996). Now, if we apply the fractionalization method, then there exists a non-zero constant $\sigma\in\mathbb{R}$ such that

$$\frac{\mathrm{d}x}{\mathrm{d}t}(t)=\frac{1}{\sigma^{1-\alpha}}\,{}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t x(t),\ \text{ with }\ 0<\alpha<1.$$

Then, we obtain the new functional to minimize:

$$J[x]=\int_b^a\sqrt{1+\left(\frac{1}{\sigma^{1-\alpha}}\,{}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t x(t)\right)^2}\,\mathrm{d}t.$$

Thus, we see that if the functional J admits a minimum at $\bar{x}$, then this function satisfies the following reduced form of the Euler–Lagrange equation (20) which is obtained by observing that L does not depend on $\bar{x}$:

$$\begin{aligned}
{}^{\mathrm{ABR}}_{\ \ t}D^\alpha_b\frac{\partial L}{\partial\, {}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t x}(t,\bar{x},{}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t\bar{x}) &=\\
&= {}^{\mathrm{ABR}}_{\ \ t}D^\alpha_b\left(\frac{\frac{1}{\sigma^{1-\alpha}}\,{}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t\bar{x}}{\sqrt{1+\left(\frac{1}{\sigma^{1-\alpha}}\,{}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t\bar{x}\right)^2}}\right)\\
&= \frac{1}{\sigma^{1-\alpha}}\,{}^{\mathrm{ABR}}_{\ \ t}D^\alpha_b\left(\frac{{}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t\bar{x}}{\sqrt{1+\left(\frac{1}{\sigma^{1-\alpha}}\,{}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t\bar{x}\right)^2}}\right)=0,
\end{aligned}$$

which implies that,

$$\frac{{}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t\bar{x}}{\sqrt{1+\left(\frac{1}{\sigma^{1-\alpha}}\,{}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t\bar{x}\right)^2}}=k,$$

with $k\in\mathbb{R}$ a constant. From this expression we obtain the following fractional linear differential equation:

$${}^{\mathrm{ABR}}_{\ \ a}D^\alpha_t\bar{x}=\frac{k\sigma^{1-\alpha}}{\sqrt{\sigma^{2(1-\alpha)}-k^2}}.$$

Now applying ${}^{\mathrm{AB}}_{\ a}I^\alpha_t$, and taking into consideration the identity (16), we obtain,

$$\begin{aligned}
\bar{x}(t) &= {}^{\mathrm{AB}}_{\ a}I^\alpha_t\left(\frac{k\sigma^{1-\alpha}}{\sqrt{\sigma^{2(1-\alpha)}-k^2}}\right)\\
&= \frac{k\sigma^{1-\alpha}}{\sqrt{\sigma^{2(1-\alpha)}-k^2}}\left(\frac{1-\alpha}{B(\alpha)}+\frac{\alpha}{B(\alpha)}\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}\,\mathrm{d}s\right),
\end{aligned}$$

that is,

$$\bar{x}(t)=\frac{k_0}{B(\alpha)}\left(1-\alpha+\frac{(t-a)^\alpha}{\Gamma(\alpha)}\right), \quad (21)$$

with

$$k_0=\frac{k\sigma^{1-\alpha}}{\sqrt{\sigma^{2(1-\alpha)}-k^2}}.$$

We observe that if we choose k_0 such that $x_a=\frac{1-\alpha}{B(\alpha)}k_0$ and x_b is defined by the equality

$$x_b=x_a\left(1+\frac{1}{1-\alpha}\frac{(b-a)^\alpha}{\Gamma(\alpha)}\right), \quad (22)$$

then

$$\bar{x}(t) = x_a \left(1 + \frac{1}{1-\alpha}\frac{(t-a)^{\alpha}}{\Gamma(\alpha)}\right). \tag{23}$$

Since x_a and x_b are arbitrary, as are the constants a and b, the relation (22) is not always satisfied, unless it is possible to choose $\alpha \in (0, 1)$ so that this relation is satisfied. In such a case, it is clear that the inequality $x_a < x_b$ must hold. On the other hand, it is observed that if we take the limit $\alpha \to 1$ in (21), then it is necessary to choose $x_a = 0$ in order to obtain the geodesic curve

$$\bar{x}(t) = \frac{x_b}{b-a}t - \frac{a x_b}{b-a}.$$

We recall that in the classical calculus of variations, the geodesic curve in the plane is described by

$$\bar{x}(t) = \frac{x_b - x_a}{b-a}t - \frac{a x_b - b x_a}{b-a}; \tag{24}$$

see Troutman (1996). Similar results can be found in Chatibi et al. (2019).

In Figure 1, we show a comparison between the solutions (23) obtained with fractional order derivatives and the classical solution (24) obtained with derivatives of integer order, using GNU Octave 7.2. As a particular case, we choose $a = 0$, $b = 5$, $x_a = 1$ and $x_b = 5$. We observe that only by choosing $\alpha = 0.5820$, we obtain an approximation of the solution that solves the problem proposed in the calculus of variations with fractional derivatives, but that such solution does not coincide with the classical solution obtained in the calculus of variations with derivatives of integer order, except at the extreme points (a, x_a) and (b, x_b).

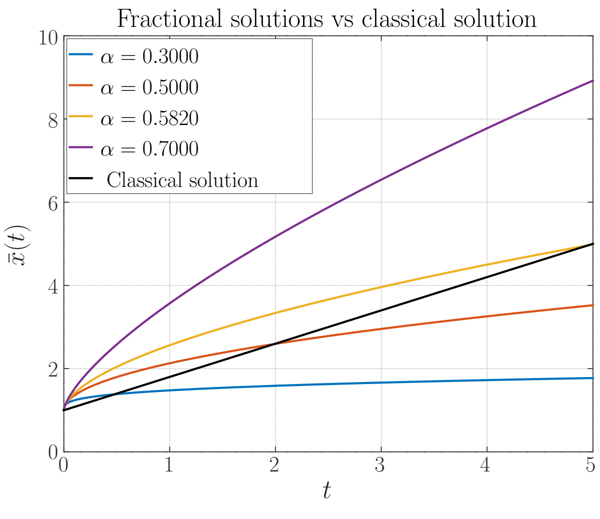

Figure 1: Comparison between fractional solutions and classical solutions obtained in the Example 1.

4. Conclusions

In this paper, we have presented the relations that exist between fractional operators in the sense of Riemann–Liouville and Atangana–Baleanu. We have also presented a variational problem that depends on the Atangana–Baleanu derivative: the problem on the determination of geodesic curves in the plane. We have established the necessary optimality conditions for the fundamental problem of the of fractional calculus of variations. Finally, it would be interesting to extend this work and study the problem on the determination of geodesic curves defined on surfaces of revolution.

References

Abdeljawad, T., Atangana, A., Gómez-Aguilar, J. F., and Jarad, F. (2019). On a more general fractional integration by parts formulae and applications. *Physica A: Statistical Mechanics and its Applications*, 536:122494. https://doi.org/10.1016/j.physa.2019.122494.

Abdeljawad, T. and Baleanu, D. (2017). Integration by parts and its applications of a new nonlocal fractional derivative with Mittag–Leffler nonsingular kernel. *Journal of Nonlinear Sciences and Applications*, 10(3):1098–1107. http://doi.org/10.22436/jnsa.010.03.20.

Agrawal, O. P. (2002). Formulation of Euler–Lagrange equations for fractional variational problems. *Journal of Mathematical Analysis and Applications*, 272(1):368–379. https://doi.org/10.1016/S0022-247X(02)00180-4.

Almeida, R., R. A. C. Ferreira, and D. F. M. Torres (2012). Isoperimetric problems of the calculus of variations with fractional derivatives. *Acta Mathematica Scientia*, 32(2):619–630. https://doi.org/10.1016/S0252-9602(12)60043-5.

Atanacković, T. M., Pilipović, S., Stanković, B., and Zorica, D. (2014). *Fractional calculus with applications in mechanics. Wave propagation, impact and variational principles*. John Wiley & Sons, Ltd, Great Britain. https://doi.org/10.1002/9781118909065.

Atangana, A. and Balenu, D. (2016). New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. *Thermal Science*, 20(2):763–769. https://doi.org/10.2298/TSCI160111018A.

Baleanu, D. (2009). Fractional variational principles in action. *Physica Scripta*, 2009(T136):014006. https://doi.org/10.1088/0031-8949/2009/T136/014006.

Baleanu, D. and Fernandez, A. (2018). On some new properties of fractional derivatives with Mittag–Leffler kernel. *Communications in Nonlinear Science and Numerical Simulation*, 59:444–462. https://doi.org/10.1016/j.cnsns.2017.12.003.

Baleanu, D. and Fernandez, A. (2019). On fractional operators and their classifications. *Mathematics*, 7(9):830. https://doi.org/10.3390/math7090830.

Bartle, R. G. (1995). *The elements of integration and Lebesgue measure*. Wiley Classics Library. John Wiley & Sons, Inc., New York. https://doi.org/10.1002/9781118164471.

Brezis, H. (2011). *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York. https://doi.org/10.1007/978-0-387-70914-7.

Caputo, M. and Fabrizio, M. (2015). A new definition of fractional derivative without singular kernel. *Progress in Fractional Differentiation and Applications*, 1(2):73–85.

Chatibi, Y., El Kinani, E. H., and Ouhadan, A. (2019). Variational calculus involving nonlocal fractional derivative with Mittag–Leffler kernel. *Chaos, Solitons & Fractals*, 118:117–121. https://doi.org/10.1016/j.chaos.2018.11.017.

Coronel-Escamilla, A., Gómez-Aguilar, J. F., Baleanu, D., Escobar-Jiménez, R. F., Olivares-Peregrino, V. H., and Abundez-Pliego, A. (2016). Formulation of Euler-Lagrange and Hamilton equations involving fractional operators with regular kernel. *Advances in Difference Equations*, 2016(1):283. https://doi.org/10.1186/s13662-016-1001-5.

Freguglia, P. and Giaquinta, M. (2016). *The early period of the calculus of variations*. Birkhäuser Cham, Switzerland. https://doi.org/10.1007/978-3-319-38945-5.

Gómez-Aguilar, J. F., Razo-Hernández, R., and Granados-Lieberman, D. (2014). A physical interpretation of fractional calculus in observables terms: analysis of the fractional time constant and the transitory response. *Revista Mexicana de Física*, 60(1):32–38.

Gorenflo, R., Kilbas, A. A., Mainardi, F., and Rogosin, S. (2020). *Mittag–Leffler functions, related topics and applications*. Springer Monographs in Mathematics. Springer, Germany, 2 edition. https://doi.org/10.1007/978-3-662-43930-2.

Lebedev, N. N. (1965). *Special functions an their applications*. Prentice-Hall, Inc., New York.

Miller, K. S. and Ross, B. (1993). *An introduction to the fractional calculus and fractional differential equations*. John Wiley & Sons, Inc., New York.

Oldham, K. B. and Spanier, J. (1974). *The fractional calculus: Theory and applications of differentiation and integration to arbitrary order*, volume 111 of *Mathematics in Science and Engineering*. Academic Press, Inc., New York.

Podlubny, I., Magin, R. L., and Trymorush, I. (2017). Niels Henrik Abel and the birth of fractional calculus. *Fractional Calculus and Applied Analysis*, 20(5):1068–1075. https://doi.org/10.1515/fca-2017-0057.

Riewe, F. (1996). Nonconservative Lagrangian and Hamiltonian mechanics. *Physical Review E*, 53(2):1890–1899. https://doi.org/10.1103/PhysRevE.53.1890.

Riewe, F. (1997). Mechanics with fractional derivatives. *Physical Review E*, 55:3581–3592. https://doi.org/10.1103/PhysRevE.55.3581.

Ross, B. (1975). A brief history and exposition of the fundamental theory of fractional calculus. In *Fractional Calculus and Its Applications*, volume 457 of *Lecture Notes in Mathematics*, pages 1–36. Springer, Berlin. https://doi.org/10.1007/BFb0067096.

Sales-Teodoro, G., Tenreiro-Machado, J. A., and Capelas de Oliveira, E. (2019). A review of definitions of fractional derivatives and other operators. *Journal of Computational Physics*, 388:195–208. https://doi.org/10.1016/j.jcp.2019.03.008.

Samko, S. G., Kilbas, A. A., and Marichev, O. I. (1993). *Fractional integrals and derivatives*. Gordon and Breach Science Publishers, Switzerland.

Troutman, J. L. (1996). *Variational calculus and optimal control*. Undergraduate Texts in Mathematics. Springer New York, NY, New ork, 2 edition. https://doi.org/10.1007/978-1-4612-0737-5.