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Robust stability problem in the mass–spring–damper system with general conformable fractional derivative
El problema de estabilidad robusta en el sistema masa–resorte–amortiguador con derivada fraccionaria conforme general

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Abstract

This paper considers a mass-spring-damper system that admits a external perturbation that is assumed to belong to a prefixed set of piecewise continuous functions. The general conformable fractional derivative is introduced in the second order differential equation that describes the dynamics of the mass-spring-damper system through the fractionalization method. The maximum deviation problem is formulated and studied in the resulting system of general conformable fractional differential equations. Using the solution of the maximum deviation problem, a maximum limit cycle is obtained, and this is used to establish a robust stability criterion for the solutions of the general conformable fractional differential equation. The robust stability criterion is obtained by considering an extension of the definition of stability under constant-acting perturbations that is used in systems of ordinary differential equations. The results obtained are illustrated numerically.

Keywords: Mass-spring-damper system, general conformable fractional derivative, reachability set, maximum deviation problem, robust stability.

Resumen

Este artículo considera un sistema masa-resorte-amortiguador que admite una perturbación externa que se supone pertenece a un conjunto prefijado de funciones continuas a trozos. La derivada fraccionaria conforme general se introduce en la ecuación diferencial de segundo orden que describe la dinámica del sistema masa-resorte-amortiguador mediante el método de fraccionalización. El problema de la máxima desviación se formula y se estudia en el sistema resultante de ecuaciones diferenciales fraccionarias conformes generales. Utilizando la solución del problema de máxima desviación, se obtiene un ciclo límite máximo, y este se utiliza para establecer un criterio de estabilidad robusta para las soluciones de la ecuación diferencial fraccionaria conforme general. El criterio de estabilidad robusta se obtiene considerando una extensión de la definición de estabilidad bajo perturbaciones de acción constante que se usa en sistemas de ecuaciones diferenciales ordinarias. Los resultados obtenidos se ilustran numéricamente.

Palabras Clave: Sistema masa-resorte-amortiguador, derivada fraccionaria conforme general, conjunto de alcanzabilidad, problema de máxima desviación, estabilidad robusta.

1. Introduction

Fractional calculus is characterized by the fact that the derivatives and integrals used have fractional order. It is known that fractional calculus has its origin in an exchange of letters between L'Hôpital and Leibniz, in which the meaning of the derivative $\frac{\mathrm{d}^n}{\mathrm{d}t^n}$ when $n = \frac{1}{2}$ was discussed, see e.g. Oldham and Spanier (1974). As a consequence of this historical event, several definitions of fractional derivatives have been introduced, among which are the definitions of Riemann–Liouville, Grunwald–Letniko, Caputo, Riesz, Riesz–Caputo, Atangana–Baleanu, Weyl, etc.; see e.g. Caponetto et al. (2014) and Sales Teodoro et al. (2019). A main characteristic of these definitions is that they do not satisfy some traditional rules of traditional calculus, for example, some of these definitions do not satisfy the product rule, the quotient rule, or the chain rule.

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In recent years, Khalil and his collaborators introduced the concept of conformable fractional derivative in Khalil et al. (2014). The conformable fractional derivative is interesting due to the number of results that have been obtained, since some problems in which conformable fractional differential equations are considered can be solved in a simpler way; see Abdeljawad (2015). Furthermore, the study on the properties and applications of the conformable fractional derivative has drawn the attention of several researchers. In particular, there is an interest in studying the main characteristics of the solutions of differential equations when conformable fractional derivatives are considered; see e.g. Souahi et al. (2017), Bayour and Torres (2017), Zhong and Wang (2018) and Öğrekçi et al. (2021). Other extensions to the concept of conformable fractional derivative have been introduced, some particular cases can be consulted in Almeida et al. (2016), Zhao and Luo (2017), Abu-Shady and Kaabar (2021) and Kajouni et al. (2021). In this paper we consider the concept of *general conformable fractional derivative* introduced in Zhao and Luo (2017).

A method sometimes used to introduce fractional derivatives into ordinary differential equations of integer order is to replace the integer derivatives with a multiple of the fractional derivatives. This procedure is called *fractionalization method*; see Rosales et al. (2011). This method of substituting integer derivatives by a multiple of the fractional derivatives in an ordinary differential equation has been widely considered in the literature, the following examples are some particular cases: in Gómez-Aguilar et al. (2012) an analysis of the solutions of a mechanical oscillator described by a fractional differential equation is carried out, in Rosales et al. (2014) a linear fractional differential equation describing the behavior of a two dimensional projectile in a resisting medium is considered, in Ebaid et al. (2017) the vertical velocity of a body in free fall is studied, and in Ortega and Rosales (2018) the conformable fractional differential equation that describes Newton's cooling law is studied.

The objective of this paper is to apply the fractionalization method to the second-order differential equation that models the dynamics of the mass-spring-damper system with external perturbations to compare the solutions of the general conformable fractional model and the classical model. Furthermore, we determine a robust stability criterion in the resulting general conformable fractional differential equation. The robust stability criterion is obtained by extending the concept of stability under constant-acting perturbations introduced by Duboshin and Malkin and which is regularly applied to differential equations with derivatives of integer order; see e.g. Elsgoltz (1970).

2. Preliminaries

This section summarizes the properties of the general conformable fractional derivative that is introduced in Zhao and Luo (2017).

Definition 1 (Zhao and Luo 2017)**.** The continuous functions $\varphi\colon [0,\infty)\times(0,1]\to\mathbb{R}$ that satisfy

- $\varphi(t,1)=1$ for all $t\geq 0$,
- $\varphi(t,\alpha)\neq 0$ for all $(t,\alpha)\in[0,\infty)\times(0,1]$,
- $\varphi(\cdot,\alpha)\neq\varphi(\cdot,\beta)$, where $\alpha\neq\beta$ and $\alpha,\beta\in(0,1]$,

and the constant function $\varphi(t,\alpha)\equiv 1$ are called *conformable fractional functions*.

Definition 2 (Zhao and Luo 2017)**.** Let $p\colon [a,b]\to\mathbb{R}$, $0\leq a<b$. The *left-sided general conformable fractional derivative* of order α of p is defined as

$$ {}_aD_t^{\alpha,\varphi}p(t)=\lim_{\epsilon\to 0}\frac{p(t+\epsilon\cdot\varphi(t,\alpha))-p(t)}{\epsilon}$$

for all $t\in(a,b)$ and $\alpha\in(0,1]$. If the limit exists, then we say that p is *left-sided α-differentiable*. Furthermore, if ${}_aD_t^{\alpha,\varphi}p(t)$ exists for all $t\in(a,b)$, then

$$ {}_aD_t^{\alpha,\varphi}p(a)=\lim_{t\to a^+}{}_aD_t^{\alpha,\varphi}p(t),\quad {}_aD_t^{\alpha,\varphi}p(b)=\lim_{t\to b^-}{}_aD_t^{\alpha,\varphi}p(t).$$

The *left-sided general conformable fractional integral* of p of order $\alpha\in(0,1]$ is defined as

$$ {}_aI_t^{\alpha,\varphi}p(t)=\int_a^t\frac{p(s)}{\varphi(s,\alpha)}\,\mathrm{d}s.$$

We say that p is *left-sided α-integrable* if the integral exists.

There is a concept of right-sided general conformable fractional derivative and right-sided general conformable fractional integral that is defined analogously; see Zhao and Luo (2017). We only consider functions that are left-sided α-differentiable and left-sided α-integrable. In what follows, we will only use the term "α-differentiable" to refer to functions that are left-sided α-differentiable. In the same way we will use the term "α-integrable" for functions that are left-sided α-integrable.

We note that for $\alpha=1$, the α-differentiation reduces to the ordinary derivative and the α-integral reduces to the ordinary integral. Furthermore, it is observed that in general if $\alpha,\beta\in(0,1]$, then

$$ {}_aD_t^{\alpha,\varphi}\,{}_aD_t^{\beta,\varphi}p(t)\neq{}_aD_t^{\alpha+\beta,\varphi}p(t),\qquad t\in(a,b).$$

The following properties of functions α-differentiable are satisfied.

Theorem 1 (Zhao and Luo 2017)**.** Let $\alpha\in(0,1]$ and let p and q be functions α-differentiable at a point $t>a$. Then

(a) If $p(t)=k$, $k\equiv$ const, then ${}_aD_t^{\alpha,\varphi}p(t)\equiv 0$.
(b) For all $k_1,k_2\in\mathbb{R}$ it holds

$$ {}_aD_t^{\alpha,\varphi}(k_1p+k_2q)(t)=k_1\,{}_aD_t^{\alpha,\varphi}p(t)+k_2\,{}_aD_t^{\alpha,\varphi}q(t).$$

(c) ${}_aD_t^{\alpha,\varphi}(p\cdot q)(t)=p(t)\cdot{}_aD_t^{\alpha,\varphi}q(t)+q(t)\cdot{}_aD_t^{\alpha,\varphi}p(t)$.
(d) ${}_aD_t^{\alpha,\varphi}\left(\dfrac{p}{q}\right)(t)=\dfrac{q(t)\cdot{}_aD_t^{\alpha,\varphi}p(t)-p(t)\cdot{}_aD_t^{\alpha,\varphi}q(t)}{q(t)^2}$.
(e) If p is differentiable, then

$$ {}_aD_t^{\alpha,\varphi}p(t)=\varphi(t,\alpha)\frac{\mathrm{d}p}{\mathrm{d}t}(t).$$

Theorem 2 (Zhao and Luo 2017)**.** If $p\colon [a,b]\to\mathbb{R}$ is α-differentiable at $t_0>a$, $\alpha\in(0,1]$, then p is continuous at t_0.

In what follows we consider a continuous and α-differentiable function $h\colon [a,b]\to\mathbb{R}$ defined as

$$h(t)=\int_a^t\frac{1}{\varphi(s,\alpha)}\,\mathrm{d}s. \tag{1}$$

We note that ${}_aD_t^{\alpha,\varphi}h(t)=1$.

The mean value theorem for general conformable fractional derivatives is as follows.

Theorem 3 (Zhao and Luo 2017)**.** Let $a > 0$ and $p\colon [a,b] \to \mathbb{R}$ be a α-differentiable function for $\alpha \in (0,1]$ at any $t \in [a,b]$. Then, there exists $c \in (a,b)$, such that

$$ {}_aD_t^{\alpha,\varphi} p(c) = \frac{p(b)-p(a)}{h(b)-h(a)}. $$

Theorem 4 (Zhao and Luo 2017)**.** If p is α-differentiable and ${}_aD_t^{\alpha,\varphi}(p)$ is bounded on $[a,b]$ and continuous on a, then p is uniformly continuous on $[a,b]$, and hence p is bounded on $[a,b]$.

The chain rule for general conformable fractional derivatives is as follows.

Theorem 5 (Zhao and Luo 2017)**.** Let $\alpha \in (0,1]$ and let p and q be functions α-differentiable at a point $t > a$. Then

$$ {}_aD_t^{\alpha,\varphi}(p \circ q)(t) = \frac{\mathrm{d}p}{\mathrm{d}t}(q(t)) \cdot {}_aD_t^{\alpha,\varphi} q(t). $$

The following two theorems correspond to versions of the fundamental theorems of calculus for general conformable fractional derivatives.

Theorem 6 (Zhao and Luo 2017)**.** If p is continuous on (a,b) then for all $t \in (a,b)$ y $\alpha \in (0,1]$ the following equality is satisfied ${}_aD_t^{\alpha,\varphi}\, {}_aI_t^{\alpha,\varphi} p(t) = p(t)$.

Theorem 7 (Zhao and Luo 2017)**.** If p is differentiable on $[a,b]$, $\alpha \in (0,1]$, then ${}_aI_t^{\alpha,\varphi}\, {}_aD_t^{\alpha,\varphi} p(t) = p(t) - p(a)$.

The version of Rolle's Theorem for general conformable fractional derivatives is as follows.

Theorem 8 (Zhao and Luo 2017)**.** Let p be a given function that satisfies

- p is α-differentiable for some $\alpha \in (0,1]$ at any $t \in [a,b]$,
- $p(a) = p(b)$.

Then, there exist $c \in (a,b)$, such that ${}_aD_t^{\alpha,\varphi} p(c) = 0$.

We remember that a function p is increasing (respectively, decreasing) if for $t_1, t_2 \in [a,b]$ with $t_1 < t_2$, then $p(t_1) < p(t_2)$ (respectively, $p(t_1) > p(t_2)$). The following theorem presents a criterion to determine when a function p is increasing or decreasing considering the derivative ${}_aD_t^{\alpha,\varphi}$.

Theorem 9. If ${}_aD_t^{\alpha,\varphi} p(t)$ exists over (a,b) and ${}_aD_t^{\alpha,\varphi} p(t) > 0$ (respectively, ${}_aD_t^{\alpha,\varphi} p(t) < 0$) for all $t \in (a,b)$, then the graph of the function p is increasing (respectively, decreasing) on (a,b).

Proof. Let $t_1, t_2 \in [a,b]$ with $t_1 < t_2$. We consider the function defined in (1) and

$$ g(t) = p(t) - p(t_1) - \frac{p(t_2)-p(t_1)}{h(t_2)-h(t_1)}(h(t) - h(t_1)). $$

The function g satisfies the conditions of Rolle's Theorem in (t_1, t_2): $g(t_1) = g(t_2)$ and g is α-differentiable for all $\alpha \in (0,1]$ and $t \in (t_1, t_2)$. Thus, there exist $t_0 \in (t_1, t_2)$ such that ${}_aD_t^{\alpha,\varphi} g(t_0) = 0$, where

$$ {}_aD_t^{\alpha,\varphi} g(t_0) = {}_aD_t^{\alpha,\varphi} p(t_0) - \frac{p(t_2)-p(t_1)}{h(t_2)-h(t_1)}\, {}_aD_t^{\alpha,\varphi} h(t_0). $$

From the above, it follows that

$$ {}_aD_t^{\alpha,\varphi} p(t_0) = \frac{p(t_2)-p(t_1)}{h(t_2)-h(t_1)}. $$

It is clear that if ${}_aD_t^{\alpha,\varphi} p(t_0) > 0$, then $p(t_2) > (t_1)$. Thus, p is increasing on (a,b). Analogously, if ${}_aD_t^{\alpha,\varphi} p(t_0) < 0$, then $p(t_2) < (t_1)$. Thus, p is decreasing on (a,b). □

Let $\boldsymbol{\gamma}\colon [a,b] \to \mathbb{R}^2$ a trajectory $\boldsymbol{\gamma}(t) = (p(t), q(t))^\top$, where p and q are assumed α-differentiable at $t \in (a,b)$. The notation $^\top$ is used to denote the transposition symbol. According to the definition of α-derivative it follows that

$$ {}_aD_t^{\alpha,\varphi} \boldsymbol{\gamma}(t) = \left({}_aD_t^{\alpha,\varphi} p(t), {}_aD_t^{\alpha,\varphi} q(t)\right)^\top, \quad t \in (a,b). $$

We conclude this section by observing that, considering the function $h(t)$ as in (1), the following general conformable fractional derivatives are valid

$$ \begin{aligned} {}_0D_t^{\alpha,\varphi} e^{\lambda h(t)} &= \lambda e^{\lambda h(t)}, \\ {}_0D_t^{\alpha,\varphi} \sin(\lambda h(t)) &= \lambda \cos(\lambda h(t)), \qquad \lambda \in \mathbb{R}. \\ {}_0D_t^{\alpha,\varphi} \cos(\lambda h(t)) &= -\lambda \sin(\lambda h(t)), \end{aligned} $$

3. Formultation of the problem

We consider the classic model of a mass-spring-damper system, which consists of a mass m that is attached to a linear spring with constant k and a linear damper with coefficient b. We assume that an external force $f(t)$ excites the dynamics of the system in such a way that the mass undergoes a displacement $x(t)$ and a speed $\frac{\mathrm{d}x}{\mathrm{d}t}(t)$. Fig. 1 shows a schematic of the described model.

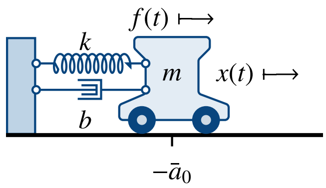

Figure 1: Mass–spring–damper system.

It is known that the second-order differential equation that describes the dynamics of the model is

$$ m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = f(t). $$

Alternatively, if we choose the natural frequency of the system $\omega = (k/m)^{1/2}$ and the damping coefficient $\mu = b/2m$, then this differential equation can be expressed as the following system of differential equations

$$ \begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= y, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -\omega^2 x - 2\mu y + u(t), \end{aligned} \tag{2} $$

where $u(t) = \frac{1}{m} f(t)$. The differential equation (2) is complemented with the initial conditions

$$ x(t)\big|_{t=0} = -\bar{a}_0, \quad y(t)\big|_{t=0} = 0, \tag{3} $$

where $\bar{a}_0 > 0$ is a constant. It is known that the dynamics of the second order differential equation depends on the possible values of the damping coefficient. In this paper, we consider the underdamped case: $0 < \mu < \omega$.

The general conformable fractional differential equation associated with (2)–(3) is obtained by the fractionalization method described in Rosales et al. (2011), which consists of introducing a new parameter σ that represents the components of the time in the general conformable fractional differential equation with a suitable dimensionality, that is, we assume that the following assignment is possible:

$$\frac{\mathrm{d}}{\mathrm{d}t} \mapsto \frac{1}{\sigma^{1-\alpha}}\, {}_0D_t^{\alpha,\varphi}, \quad \alpha \in (0,1]. \tag{4}$$

According to the development exposed in Cruz-Duarte et al. (2020), we can choose $\sigma^{1-\alpha} = \omega^{\alpha-1}$, with which, the general conformable fractional differential equation associated with (2) has the following form

$$\omega^{1-\alpha}\, {}_0D_t^{\alpha,\varphi}(\omega^{1-\alpha}\, {}_0D_t^{\alpha,\varphi} x) + 2\mu\omega^{1-\alpha}\, {}_0D_t^{\alpha,\varphi} x + \omega^2 x = u(t).$$

Considering the new variable $y = \omega^{1-\alpha}\, {}_0D_t^{\alpha,\varphi} x$, we obtain the system of general conformable fractional differential equations

$$\begin{aligned} {}_0D_t^{\alpha,\varphi} x &= \omega^{\alpha-1} y, \\ {}_0D_t^{\alpha,\varphi} y &= -\omega^{\alpha+1} x - 2\mu\omega^{\alpha-1} y + \omega^{\alpha-1} u(t). \end{aligned} \tag{5}$$

The initial conditions for (5) are

$$x(t)\big|_{t=0} = -\bar{a}_0, \quad y(t)\big|_{t=0} = 0. \tag{6}$$

If we assume that the only knowledge about the external perturbation is that it is a bounded function: $|u(t)| \leq \delta$, we can assume that this function represents an external perturbation that belongs to the set

$$u(t) \in \mathcal{U} = \{u(t) \in PC(\mathbb{R}) : |u(t)| \leq \delta\},$$

where $PC(\mathbb{R})$ denotes the set of piecewise continuous functions defined on $\mathbb{R}$ and $\delta > 0$ is a constant.

The reachability set Q of the system of general conformable fractional differential equations (5)–(6) with external perturbations $u(t) \in \mathcal{U}$ is defined as

$$\begin{aligned} Q = \{&(x(t), y(t))^\top \in \mathbb{R}^2 : \\ &(x(t), y(t))^\top \text{ satisfies (5)–(6) for some } u(t) \in \mathcal{U}\}. \end{aligned}$$

The problem on the determination of the reachability set Q has been extensively studied for the initial value problem (2)–(3) with external perturbations $u(t) \in \mathcal{U}$; see e.g. Kurzhanski and Varaiya (2014). A method that has generally been used consists of using the solution of the maximum deviation problem formulated by Bulgakov; see Elishakoff and Ohsaki (2010). In this paper, the main properties of the reachability set Q of the system of general conformable fractional differential equations (5)–(6) with external perturbations $u(t) \in \mathcal{U}$ are determined.

4. The maximum deviation problem

We consider a point $\boldsymbol{p}_0 = (-\bar{a}_0, 0)^\top$ with $\bar{a}_0 > 0$ a fixed constant value, and then solve the maximum deviation problem following the geometric argument given in Formalskii (2015) for systems of differential equations of integer order.

In the positive half-plane we consider the set of trajectories $\boldsymbol{\gamma}_u(t) = (x(t), y(t))^\top$, $t \geq 0$, that correspond to an external perturbation $u(t) \in \mathcal{U}$ and that have $\boldsymbol{p}_0$ as their starting point. The set of these curves covers a set $F(\boldsymbol{p}_0)$ of points in the plane that form a *reachable set*; see Fig. 2.[1] The boundary of this set is described by two trajectories $\boldsymbol{\gamma}_{\underline{u}}(t)$ and $\boldsymbol{\gamma}_{\bar{u}}(t)$ that are associated with the external perturbations $\underline{u}(t), \bar{u}(t) \in \mathcal{U}$, where $\bar{u}(t)$ is obtained from the following geometric argument that is valid due to the properties of the general conformable fractional derivative:

$$\begin{aligned} \bar{u}(t) &= \arg\left\{\arg\max_{u(t)\in\mathcal{U}} {}_0D_t^{\alpha,\varphi}\boldsymbol{\gamma}_u(t)\right\} \\ &= \arg\left\{\max_{u(t)\in\mathcal{U}} \frac{{}_0D_t^{\alpha,\varphi} y(t)}{{}_0D_t^{\alpha,\varphi} x(t)}\right\} \\ &= \arg\left\{\max_{u(t)\in\mathcal{U}}\left\{-\omega^2\frac{x(t)}{y(t)} - 2\mu\omega + \frac{u(t)}{y(t)}\right\}\right\} \\ &= \delta\,\mathrm{sign}\, y(t). \end{aligned} \tag{7}$$

A similar procedure is used to show that $\underline{u}(t) = -\delta\,\mathrm{sign}\, y(t)$.

After substituting the function (7) into (5), considering the initial conditions (6), and using the procedures described in Bayour and Torres (2017) or in Younas et al. (2020), we obtain

$$\boldsymbol{\gamma}_{\bar{u}}(t) = (x_{\bar{a}_0}(t), y_{\bar{a}_0}(t))^\top, \quad t \in (0, t_1),$$

whose coordinates are defined by

$$\begin{aligned} x_{\bar{a}_0}(t) &= \frac{\delta}{\omega^2} - \left(\bar{a}_0 + \frac{\delta}{\omega^2}\right) e^{-\mu\omega^{\alpha-1}h(t)} \cdot \\ &\quad \cdot \left(\cos(\beta\omega^{\alpha-1}h(t)) + \frac{\mu}{\beta}\sin(\beta\omega^{\alpha-1}h(t))\right), \\ y_{\bar{a}_0}(t) &= \frac{\omega^2}{\beta}\left(\bar{a}_0 + \frac{\delta}{\omega^2}\right) e^{-\mu\omega^{\alpha-1}h(t)} \sin(\beta\omega^{\alpha-1}h(t)), \end{aligned}$$

where $\beta = (\omega^2 - \mu^2)^{1/2}$, $h(t)$ is the function defined in (1), and $t_1 > 0$ is the instant for which $y_{\bar{a}_0}(t) \geq 0$ for all $t \in (0, t_1)$. From the previous expressions, the value $\bar{a}_1 = x_{\bar{a}_0}(t_1)$ is obtained and is called *first maximum deviation of the coordinate x*, and the instant $t_1 > 0$ is determined from the condition $y_{\bar{a}_0}(t_1) = 0$. In a such case,

$$\bar{a}_1 = A\bar{a}_0 + \frac{\delta}{\omega^2}(1 + A), \tag{8}$$

t_1 is the solution of the equation $\beta\omega^{\alpha-1}h(t_1) = \pi$, and

$$A = \exp\left(-\frac{\pi\mu}{\beta}\right).$$

Since $F(\boldsymbol{p}_0)$ contains the trajectories $\boldsymbol{\gamma}_u(t)$ that are associated with each choice $u(t) \in \mathcal{U}$, the *first maximum deviation* $\bar{b}_1$ *of the coordinate* y is obtained from the condition $\bar{b}_1 = y_{\bar{a}_0}(\hat{t}_1)$, where the instant $\hat{t}_1 \in (0, t_1)$ is determined from the equation

[1]The set $F(\boldsymbol{p}_0)$ is also called *integral funnel*, see Butkovskiy (1990).

${}_0D_t^{\alpha,\varphi}\boldsymbol{\gamma}_{\bar{u}}(\hat{t}_1) = k\boldsymbol{e}_1$, where $k > 0$ and $\boldsymbol{e}_1^\top = (1, 0)$ is the first canonical vector of $\mathbb{R}^2$, therefore, the following equality must be satisfied ${}_0D_t^{\alpha,\varphi}y_{\bar{a}_0}(\hat{t}_1) = 0$. From the solution of the system of general conformable fractional differential equations (5), we obtain that

$$\bar{b}_1 = \omega\left(\bar{a}_0 + \frac{\delta}{\omega^2}\right)B, \tag{9}$$

$\hat{t}_1$ is the solution of the equation $\beta\omega^{\alpha-1}h(\hat{t}_1) = \frac{\pi}{2} - \arctan\frac{\mu}{\beta}$, and

$$B = \exp\left(-\frac{\mu}{\beta}\left(\frac{\pi}{2} - \arctan\frac{\mu}{\beta}\right)\right).$$

Fig. 2 illustrates this geometric construction.

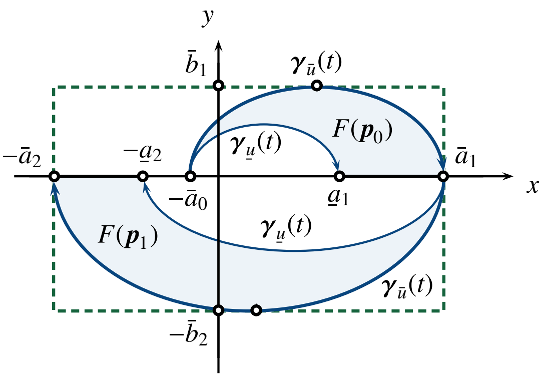

Figure 2: Geometric interpretation of the maximum deviation problem.

We observe that $\bar{a}_1$ and $\bar{b}_1$ do not depend on the order $\alpha \in (0, 1]$ of the general conformal fractional derivative in (5) and that the instants t_1 and $\hat{t}_1$ does depend on $\alpha \in (0, 1]$.

On the other hand, if in the system of general conformable fractional differential equations (5) we substitute the external perturbation $\underline{u}(t) = -\delta \operatorname{sign} y(t)$ and denote the corresponding solution by $\boldsymbol{\gamma}_{\underline{u}}(t)$, then at time t_1 the parameter $\underline{a}_1$ is obtained from the condition $\boldsymbol{\gamma}_{\underline{u}}(t_1) = (\underline{a}_1, 0)^\top$. The value $\underline{a}_1$ is called *first minimum deviation of coordinate* x; see Fig. 2. This allows us to observe that if we choose $\boldsymbol{p}(s) = (1-\lambda)\boldsymbol{\gamma}_{\underline{u}}(s) + \lambda\boldsymbol{\gamma}_{\bar{u}}(s)$ with $\lambda \in [0, 1]$ and $s \in [0, t_1]$, then the trajectory $\boldsymbol{\gamma}_{\hat{u}}(t)$ associated with the external perturbation $\hat{u}(t) = (1-\lambda)\underline{u}(t) + \lambda\bar{u}(t) \in \mathcal{U}$ satisfies $\boldsymbol{\gamma}_{\hat{u}}(s) = \boldsymbol{p}(s)$, that is, $\boldsymbol{p}(s) \in F(\boldsymbol{p}_0)$. From this it follows that $F(\boldsymbol{p}_0)$ is a simply connected set.

The argument used to construct $F(\boldsymbol{p}_0)$ is symmetric, in the sense that if we fix the point $\boldsymbol{p}_1 = (\bar{a}_1, 0)^\top$, where $\bar{a}_1$ is defined in (8), then we can construct the reachable set $F(\boldsymbol{p}_1)$ with the property that the lower boundary of this set is described by the extension of the trajectory $\boldsymbol{\gamma}_{\bar{u}}(t)$, whose components are solutions of (5) defined in $(0, t_2)$, where $t_2 > t_1$; see Fig. 2. In this case, we obtain the trayectory

$$\boldsymbol{\gamma}_{\bar{u}}(t) = \begin{cases} \left(x_{\bar{a}_0}(t), y_{\bar{a}_0}(t)\right)^\top, & t \in (0, t_1), \\ \left(x_{\bar{a}_1}(t-t_1), y_{\bar{a}_1}(t-t_1)\right)^\top, & t \in (t_1, t_2), \end{cases}$$

whose coordinates $x_{\bar{a}_1}(t)$ and $y_{\bar{a}_1}(t)$ are defined by

$$x_{\bar{a}_1}(t) = -\frac{\delta}{\omega^2} + \left(\bar{a}_1 + \frac{\delta}{\omega^2}\right)e^{-\mu\omega^{\alpha-1}h(t)} \cdot \left(\cos(\beta\omega^{\alpha-1}h(t)) + \frac{\mu}{\beta}\sin(\beta\omega^{\alpha-1}h(t))\right),$$

$$y_{\bar{a}_1}(t) = -\frac{\omega^2}{\beta}\left(\bar{a}_1 + \frac{\delta}{\omega^2}\right)e^{-\mu\omega^{\alpha-1}h(t)}\sin(\beta\omega^{\alpha-1}h(t)),$$

where $t_2 > t_1$ is the instant for which $y_{\bar{a}_1}(t-t_1) \leq 0$ for all $t \in (t_1, t_2)$. The external perturbation is again described by $\bar{u}(t) = \delta \operatorname{sign} y(t)$ and $t_2 = 2t_1$. Furthermore, we obtain from the extension of the trajectory in the interval (t_1, t_2) that the second maximum deviation of the coordinate x, which satisfies the relation $-\bar{a}_2 = x_{\bar{a}_1}(t_2 - t_1)$, is described by

$$\bar{a}_2 = A\bar{a}_1 + \frac{\delta}{\omega^2}(1 + A), \tag{10}$$

where t_2 is the solution of the equation $\beta\omega^{\alpha-1}h(t_2 - t_1) = \pi$, and the second maximum deviation $\bar{b}_2$ of the coordinate y, which satisfies the relation $-\bar{b}_2 = y_{\bar{a}_1}(\hat{t}_2 - t_1)$, is described by

$$\bar{b}_2 = \omega\left(\bar{a}_1 + \frac{\delta}{\omega^2}\right)B, \tag{11}$$

and where $\hat{t}_2$ is obtained from the equation ${}_0D_t^{\alpha,\varphi}y_{\bar{a}_1}(\hat{t}_2 - t_1) = 0$, which is equivalent to $\beta\omega^{\alpha-1}h(\hat{t}_2 - t_1) = \frac{\pi}{2} - \arctan\frac{\mu}{\beta}$. We conclude that $\hat{t}_2 = 2\hat{t}_1$. Furthermore, it can also be verified that $F(\boldsymbol{p}_1)$ is a simply connected set.

We note once again that $\bar{a}_2$ and $\bar{b}_2$ do not depend on the order $\alpha \in (0, 1]$ of the general conformable fractional derivative in (5) while the instants t_2 and $\hat{t}_2$ do depend on it.

The procedure that was used to obtain the maximum deviations (8) and (9), as well as (10) and (11), can be repeated recursively in order to obtain two sequences of maximum deviations $\{\bar{a}_k\}_{k\in\mathbb{N}}$ and $\{\bar{b}_k\}_{k\in\mathbb{N}}$ that depend on the choice $\bar{a}_0 > 0$ defined by

$$\left.\begin{aligned} \bar{a}_k &= A\bar{a}_{k-1} + \frac{\delta}{\omega^2}(1 + A), \\ \bar{b}_k &= \omega\left(\bar{a}_{k-1} + \frac{\delta}{\omega^2}\right)B, \end{aligned}\right\} \quad k \geq 1, \tag{12}$$

which are obtained from the trajectory $\boldsymbol{\gamma}_{\bar{u}}(t)$ described by

$$\boldsymbol{\gamma}_{\bar{u}}(t) = \begin{cases} \left(x_{\bar{a}_0}(t), y_{\bar{a}_0}(t)\right)^\top, & t \in (0, t_1), \\ \left(x_{\bar{a}_1}(t-t_1), y_{\bar{a}_1}(t-t_1)\right)^\top, & t \in (t_1, 2t_1), \\ \cdots & \cdots \\ \left(x_{\bar{a}_k}(t-t_k), y_{\bar{a}_k}(t-t_k)\right)^\top, & t \in (t_k, 2t_k). \end{cases}$$

According to Elaydi (2005), if a recurrence relation defined by $c_{k+1} = g(c_k)$ is such that g has an equilibrium point $c^* \in \mathbb{R}$, and if $|\frac{dg}{d\xi}(\xi)| < 1$ for all $\xi \in \mathbb{R}$, then the equilibrium point of g is asymptotically stable, that is, $c^* = \lim_{k\to\infty} c_k$. Therefore, we have the following limits for (12):

$$\begin{aligned} \bar{a}_* &= \lim_{k\to\infty}\bar{a}_k = \frac{\delta}{\omega^2}\frac{1+A}{1-A} = \frac{\delta}{\omega^2}\coth\frac{\pi\mu}{2\beta}, \\ \bar{b}_* &= \lim_{k\to\infty}\bar{b}_k = \frac{\delta}{\omega}\left(1 + \coth\frac{\pi\mu}{2\beta}\right)\cdot \\ &\qquad \cdot\exp\left(-\frac{\mu}{\beta}\left(\frac{\pi}{2} - \arctan\frac{\mu}{\beta}\right)\right). \end{aligned} \tag{13}$$

The parameters $\bar{a}_*$ and $\bar{b}_*$ are known as the *maximum deviations* of the corresponding coordinates x and y. We observe that the parameters $\bar{a}_*$ and $\bar{b}_*$ do not depend on the order $\alpha \in (0, 1]$ of the general conformable fractional derivative in (5). Furthermore, it can be verified that if the following variation of the initial condition $\bar{a}_0 = \bar{a}_* \pm \delta a$ is chosen in (12), where $\delta a > 0$, then the corresponding sequences of maximum deviations also converge respectively to $\bar{a}_*$ and $\bar{b}_*$.

The convergence of the discrete dynamical systems in (13) allows us to conclude that the system of general conformable fractional differential equations (5) has a unique limit cycle $\boldsymbol{\gamma}^*_{\bar{u}}(t) = (x_{\bar{a}_*}(t), y_{\bar{a}_*}(t))^\top$, $t \in (0, t_1)$, whose coordinates are defined by

$$x_{\bar{a}_*}(t) = \pm\frac{\delta}{\omega^2} \mp \left(\bar{a}_* + \frac{\delta}{\omega^2}\right) e^{-\mu\omega^{\alpha-1}h(t)}\cdot \left(\cos(\beta\omega^{\alpha-1}h(t)) + \frac{\mu}{\beta}\sin(\beta\omega^{\alpha-1}h(t))\right),$$
$$y_{\bar{a}_*}(t) = \pm\frac{\omega^2}{\beta}\left(\bar{a}_* + \frac{\delta}{\omega^2}\right) e^{-\mu\omega^{\alpha-1}h(t)} \sin(\beta\omega^{\alpha-1}h(t));$$

see Fig. 3. This limit cycle is obtained under the effect of the perturbation $\bar{u}(t) = \delta\,\mathrm{sign}\,y(t)$ in the system of general conformable fractional differential equations (5). The limit cycle $\boldsymbol{\gamma}^*_{\bar{u}}(t)$ is asymptotically orbitally stable, according to the well-known Poicaré map; see e.g. (Fradkov and Pogromsky, 1998, Theorem 3.15). The perturbation $\bar{u}(t)$ is known as the *worst external perturbation* and the limit cycle $\boldsymbol{\gamma}^*_{\bar{u}}(t)$ is known as the *maximum limit cycle*. We note that if $\bar{a}_0 > 0$, then the maximum limit cycle $\boldsymbol{\gamma}^*_{\bar{u}}(t)$ is also obtained if an initial condition of the form $\boldsymbol{p}_0 = (\bar{a}_0, 0)^\top$ is considered.

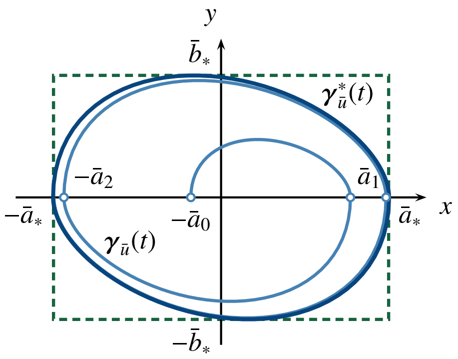

Figure 3: Geometric interpretation of the maximum limit cycle of the system of general conformable fractional differential equations (5).

We observe that the maximum limit cycle of the system of general conformable fractional differential equation (5) is invariant with respect to the order $\alpha \in (0, 1]$, that is, we have the same maximum limit cycle for each choice $\alpha \in (0, 1]$. This follows from observing that if $\tau = \omega^{\alpha-1}h(t)$ with $t \in (0, t_1)$, then the maximum limit cycle $\boldsymbol{\gamma}^*_{\bar{u}}(\tau) = (x_{\bar{a}_*}(\tau), y_{\bar{a}_*}(\tau))^\top$ has the following coordinates

$$x_{\bar{a}_*}(\tau) = \pm\frac{\delta}{\omega^2} \mp \left(\bar{a}_* + \frac{\delta}{\omega^2}\right) e^{-\mu\tau}\left(\cos(\beta\tau) + \frac{\mu}{\beta}\sin(\beta\tau)\right),$$
$$y_{\bar{a}_*}(\tau) = \pm\frac{\omega^2}{\beta}\left(\bar{a}_* + \frac{\delta}{\omega^2}\right) e^{-\mu\tau}\sin(\beta\tau). \tag{14}$$

This result is justified by observing that in the system of general conformable fractional differential equations (5), which is obtained from (4), the general conformable fractional derivative with respect to x and with respect to y, are exactly a multiple of the derivative of integer order multiplied by the function $\varphi(t, \alpha)$.

We finally note that the coordinates in (14) correspond to the coordinates of the maximum limite cycle $\boldsymbol{\gamma}^*_{\bar{u}}(t)$ as $\alpha \to 1$. The limit cycle described by (14) coincides with that described in Aleksandrov et al. (2021) for a second order differential equation (2) with initial conditions (3).

The above can be summarized as follows:

Theorem 10. The system of general conformable fractional differential equations (5) has a unique maximum limit cycle $\boldsymbol{\gamma}^*_{\bar{u}}(\tau)$ which is uniquely described regardless of the order α in $(0, 1]$. The maximum limit cycle is obtained under the effect of the worst external perturbation $\bar{u}(t) = \delta\,\mathrm{sign}\,y(t)$.

5. Robust stability criterion

The importance of the maximum limit cycle $\boldsymbol{\gamma}^*_{\bar{u}}(t)$ in the robust stability criteria for differential equations of integer order is known in the literature, see e.g. Aleksandrov et al. (2010, 2021). In view of Theorem 10, a robust stability criterion can now be established for the case of the conformable fractional differential equation (5) with external perturbations $u(t) \in \mathcal{U}$. In this section we establish a criterion based on an extension of the definition of stability under constant-acting perturbations introduced by Duboshin and Malkin for systems of differential equations of integer order; see e.g. Elsgoltz (1970). With this objective, we consider the following norm for the solutions of (5):

$$\|\boldsymbol{\gamma}_u\| = \max\left\{\sup_{t\in[0,\infty)} |x(t)|, \sup_{t\in[0,\infty)} |y(t)|\right\},$$

assuming initial conditions (6) of the form $\boldsymbol{\gamma}_u(0) = \boldsymbol{p}$, where $\boldsymbol{p} = (\bar{a}_0, 0)^\top$ and $|\bar{a}_0| \le \bar{a}_*$. Furthermore, we consider the norm $\|\boldsymbol{p}\|_\infty = \max\{|p_1|, |p_2|\}$ for $\boldsymbol{p} = (p_1, p_2)^\top \in \mathbb{R}^2$.

On the other hand, if in the system of general conformable fractional differential equations (5) there is no external perturbation and the initial conditions are zero, that is, if $u_0(t) \equiv 0$ and $\bar{a}_0 = 0$, then the corresponding solution is the trivial solution: $\boldsymbol{\gamma}_{u_0}(t) \equiv \mathbf{0}$. Such a solution is called *unperturbed*. We consider the following concept:

Definition 3. The unperturbed solution $\boldsymbol{\gamma}_{u_0}(t)$ of the system of general conformable fractional differential equations (5) is robustly stable with respect to external perturbations $u(t) \in \mathcal{U}$ and initial conditions $\boldsymbol{\gamma}_u(0) = \boldsymbol{p}$, if for all $\epsilon > 0$ exist $\eta_1(\epsilon) > 0$ and $\eta_2(\epsilon) > 0$ such that under the conditions: $\|\boldsymbol{p}\|_\infty \le |\bar{a}_0| < \eta_1(\epsilon)$ and $|u(t)| \le \delta < \eta_2(\epsilon)$ for all $t \ge 0$, any solution of the system of conformable fractional differential equation (5) satisfies the inequality $\|\boldsymbol{\gamma}_u\| < \epsilon$.

According to the construction of the maximum deviations $\bar{a}_*$ and $\bar{b}_*$, we observe that the maximum limit cycle $\boldsymbol{\gamma}^*_{\bar{u}}(t)$ has the tangent lines $x = \pm\bar{a}_*$ and $y = \pm\bar{b}_*$, from which it follows that for any other choice $u(t) \in \mathcal{U}$, the corresponding trajectory satisfies the inclusion

$$\boldsymbol{\gamma}_u(t) \subset [-\bar{a}_*, \bar{a}_*] \times [-\bar{b}_*, \bar{b}_*], \quad t \ge 0.$$

Therefore, the estimate $\|\boldsymbol{\gamma}_u\| < \epsilon$ is satisfied whenever $\max\{\bar{a}_*, \bar{b}_*\} < \epsilon$ is satisfied, that is, whenever

$$\delta < \eta_2(\epsilon) = \frac{\epsilon}{\max\{\hat{p}, \hat{q}\}}, \tag{15}$$

where

$$\hat{p} = \frac{1}{\omega^2}\coth\frac{\pi\mu}{2\beta},$$
$$\hat{q} = \frac{1}{\omega}\left(1 + \coth\frac{\pi\mu}{2\beta}\right)\exp\left(-\frac{\mu}{\beta}\left(\frac{\pi}{2} - \arctan\frac{\mu}{\beta}\right)\right).$$

Evidently,

$$\eta_1(\epsilon) = \delta\hat{p}, \tag{16}$$

and therefore, $\eta_1(\epsilon) < \hat{p}\eta_2(\epsilon)$.

The following example shows the validity of the robust stability criterion.

Example 1. We assume that the parameters in the system of conformable fractional differential equations (5) are $\mu = 0.2$ and $\omega = 1.0$. With these parameters we have $\hat{p} \approx 3.3507$ and $\hat{q} \approx 3.2897$. If we consider the problem of finding solutions of (5) in such a way that they satisfy $\|\boldsymbol{\gamma}_u(t)\| < 1$, then from (15) and (16) it follows that it suffices to choose $\eta_1(\epsilon) = 0.9000$ and $\eta_2(\epsilon) = 0.2985$. We choose $\bar{a}_0 = -0.5$ and $\delta = \frac{9}{10}\eta_2(\epsilon)$ and, as a particular case, the cases $\alpha = 0.4$, $\alpha = 0.7$ and $\alpha = 1.0$ are considered. Fig. 4 shows the trajectories of the functions $x(t)$ and $y(t)$ in which the same number of maximum deviations have been determined for $t \in [0, 35]$. The corresponding limit cycles are observed in Fig. 5. We see that the corresponding trajectories in the phase plane coincide.

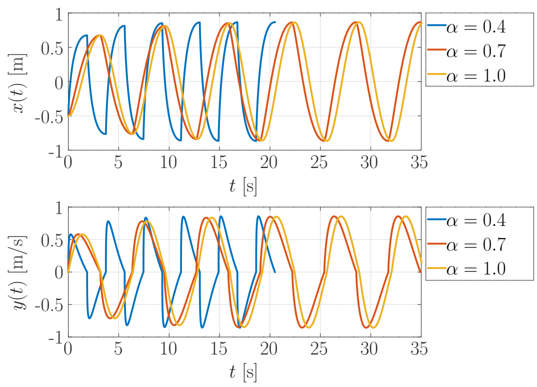

Figure 4: Displacements $x(t)$ and $y(t)$ of the system of conformable fractional differential equations.

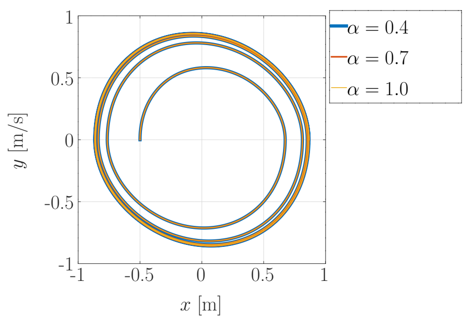

Figure 5: Trajectories of the system of conformable fractional differential equations that converge to the maximum limit cycle.

6. Conclusion

With the help of the solution of the maximum deviation problem, a maximum limit cycle for the conformal fractional differential equation has been constructed. The construction of the maximum limit cycle has allowed establishing a robust stability criterion for the unperturbed solution of the conformable fractional differential equation. It has been proved that this maximum limit cycle and the corresponding robust stability criterion do not depend on the fractional order of the conformable fractional derivative.

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