DESDE 2013
https://repository.uaeh.edu.mx/revistas/index.php/icbi/issue/archive
Pädi Boletín Científico de Ciencias Básicas e Ingenierías del ICBI

ISSN: 2007-6363

Publicación Semestral Pädi Vol. 14 No. 28 (2027) 1-13

The zeta function and Riemann hypothesis: a numerical approach

La función z y la hipótesis de Riemann: un enfoque numérico

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Resumen

En este trabajo se revisa la función Zeta y la hipótesis de Riemann. En primer lugar, se presenta un análisis exhaustivo de la función, incluyendo el estudio de sus ceros, los cuales se clasifican en triviales y no triviales. La hipótesis de Riemann se plantea en términos de la localización de estos últimos, y se discuten algunas implicaciones de su posible demostración. Posteriormente, se examina la propuesta establecida por Guilherme França y André LeClair, esto es, una aproximación numérica que busca sustentar la hipótesis al demostrar que el producto de Euler es válido para valores $R(s) > 1/2$, es decir, en la franja crítica. Aquí comparamos numéricamente las aproximaciones de la función Zeta mediante el producto de Euler y la función eta, destacando la convergencia en el rango crítico R(s)>1/2 para varios valores de *t*.

Palabras Clave: Función Z de Riemann, Hipótesis de Riemann, Ceros no triviales, Producto de Euler, Aproximación numérica.

Abstract

This work reviews the Riemann zeta function and the Riemann hypothesis. First, a comprehensive analysis of the function is presented, including the study of its zeros, which are classified as trivial and nontrivial. The Riemann hypothesis is formulated in terms of the location of the nontrivial zeros, and some implications of its potential proof are discussed. Subsequently, the proposal by Guilherme França and André LeClair is examined, namely, a numerical approach that aims to support the hypothesis by showing that Euler's product remains valid for values $R(s) > 1/2$, that is, within the critical strip. Here, we numerically compare Z function approximations through the Euler product and eta function, highlighting the convergence in the critical range $R(s) > 1/2$ for several values of t.

Keywords: Riemann Zeta function, Riemann hypothesis, Nontrivial zeros, Euler product, Numerical approach.

1. Introduction

The Zeta function $\zeta(s)$ is defined on the whole complex plane with just one pole in $s = 1$. Its functional form clearly shows that has zeros at negative even numbers. However, also it gets annul for other zeros known as non-trivial zeros. The Riemann hypothesis establishes that the real part of every non-trivial zero is equal to 1/2, that is, into the complex plane, all non-trivial zeros lie in a line located at $R(s) = 1/2$ named as the critical line (Borwein et al., 2008; Derbyshire, 2003; Edwards, 1974; McGowan, 2024).

There have been several attempts to prove or reject The Riemann hypothesis. In fact, many mathematicians have obtained zeros where the hypothesis says exactly what they should be. Bernhard Riemann was the first person who calculated three zeros (Poussin, 1897), after him plenty calculations have been performed, for instance, the great English mathematician Alan Turing found more than one thousand and so on through these years until now that we have billions of those zeros (Turing, 1953).

However, we cannot take for granted that the Riemann hypothesis is true, even having this overwhelming numerical

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Historial del manuscrito: recibido el 08/08/2025, última versión-revisada recibida el 23/03/2026, aceptado el 17/08/2026, en línea (postprint) desde el 08/09/2026, publicado el 05/01/2026. **DOI:** https://doi.org/10.29057/icbi.v14i28.15658

evidence, since that is not how mathematics works. In fact, finding a single point outside the critical line would imply ruling it out.

In this work, we expose the Guilherme França and André LeClaire perspective in the paper "On the validity of the Euler product inside the critical strip" (França and LeClair, 2014), where the Riemann hypothesis is approached from a numerical perspective with the aim of approximating the critical line from the left and right. To accomplish it, we present the mathematical formalism employed by the authors, and show some plots of the numerical results.

At this point, a natural question arises: why is so significance to prove the Riemann hypothesis to be true? Firstly, we must understand the importance of the prime numbers. The fundamental theorem of arithmetic states that every integer greater than one can be expressed as the product of one or more prime numbers, leading us to conclude these are the building blocks of all natural numbers, therefore, they are so important, and is vital for mathematics to understand their behavior. For this purpose, we define the function $\pi(x)$, which counts the prime numbers less than or equal to x. For example:

$\pi(2) = 1\ (the\ only\ prime\ less\ than\ or\ equal\ to\ 2\!:\ 2)\ \pi(3) =$

It has been demonstrated that for very large values of x, the function $\pi(x)$, behaves approximately as $x/\log log\,(x)$. This behavior is known as the prime number theorem (Apostol, 2013; Poussin, 1897; Gauss, 1849; Hadamard, 1893; Ingham, 1990). Over time, better approximations have been found; for example, the function $\pi(x)$, can also be expressed in terms of the function $Li(x)$ plus corrections related to the zeros of the Zeta function (Apostol, 2013). Therefore, if the Riemann hypothesis is proven to be true, we could bound the value of $Li(x)$ and would have a significant improvement in the estimation of the asymptotic values of $\pi(x)$, leading to a better understanding of the prime numbers distribution.

This article is divided as follows: in Section 2 we define the Zeta function, and show its extension to the real complex plane and then to the whole complex plane. We show what the Riemann hypothesis states and what are the implications of proving it to be true in Section 3. We continue with Guilherme Françca and André LeClair's numerical approach to show that the hypothesis is true, given in Section 4. In Section 5 we show numerical verification, and finally, in Section 6 conclusions are provided.

2. The Zeta Function

The Riemann Zeta function denoted by $\zeta(s)$, is a function of complex variable, $f\colon C \rightarrow C$. In order to have an historical overview on how it was studied, let us begin with the famous *Basel problem* (Ayoub, 1974), which asks for the precise summation of the reciprocals of the squares of the natural numbers, *i.e.*, which seeks for the exact sum of the series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \qquad (1)$$

The great mathematician Leonhard Euler in 1735 found the exact value in a very clever way. Let's review his method. Euler based his research on two facts:

[1]. The Taylor expansion of $sin(x)$ which is given by

$$\sin sin\,(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \qquad (2)$$

[2]. The factorization of a polynomial into factors of its roots, that is, if a polynomial $P(x)$ has non-zero roots $a_1,\ a_2, \ldots, a_n$, then one can write it as

$$P(x) = k\left(1 - \frac{x}{a_1}\right)\left(1 - \frac{x}{a_2}\right)\ldots\left(1 - \frac{x}{a_n}\right). \qquad (3)$$

where k is a constant. Starting from these statements, we can operate with them. Firstly, dividing equation (2) by x, we have:

$$\frac{\sin sin\,(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots \qquad (4)$$

On the other hand, roots of $\sin sin\,(x)/x$ are precisely in $x = n\pi$ where $n = \pm 1,\ \pm 2,\ \pm 3, \ldots$, and so according to equation (3), we can write:

$$\frac{\sin sin\,(x)}{x} = \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right)$$

$$\frac{\sin sin\,(x)}{x} = 1 - \frac{x^2}{\pi^2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right) + \ldots \qquad (5)$$

By comparing equations (4) and (5), we conclude that the corresponding coefficients are:

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{3!}, \qquad (6)$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}. \qquad (7)$$

We can continue and get the general form as can be written as:

$$\zeta(r) = \sum_{n=1}^{\infty}\frac{1}{n^r},\ r \epsilon R, \qquad (8)$$

where the case $r = 1$ is the well known divergent harmonic series. Then, we can conclude that equation (8) has exact values when $r > 1$. Nevertheless, it is common find on books expression such as $\zeta(0) = -\ 1/2$ or $\zeta(-\ 1) = -\ 1/2$. Moreover, this function with complex numbers instead reals $r \rightarrow s,\ \ s \epsilon C$, has an exact value as well. How is possible? Well, equation (8) can be extended to whole complex plane where the results mentioned before will makes sense; we will review later how to obtain such extension valid in C.

It is worth to mention that the expression given in equation (8) is sometimes called the Z function but is not completely correct, since just its analytic continuation to the whole complex plane can be named like that. However, seeking clarity, through this work we call it Z function and we are going to use s as its argument, specifying whether it is real or complex in every case.

Leonard Euler not only gave the exact summation of equation (8) evaluated on $s = 2$. He also proved the close relationship between the Zeta function and prime numbers known as "The Euler Product Formula", a key piece for this work, which is determined by:

$$\zeta(s) = \prod_{primes} \frac{1}{1-p^{-s}}, \quad for\ all\ s \in C\ with\ R(s) > 1 \tag{9}$$

To prove it we follow the ideas contained in (Amr, 2016), for all $s \in C$ with $R(s) > 1$, let's start by expanding the summation given in equation (8),

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \ldots \tag{10}$$

By multiplying it for $1/2^s$, we have:

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \ldots \tag{11}$$

If we subtract (11) from (10), the result is:

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \ldots \tag{12}$$

Now, we can multiply the equation (12) by $1/3^s$, that is:

$$\frac{1}{3^s}\left(1 - \frac{1}{2^s}\right)\zeta(s) = \frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \ldots, \tag{13}$$

and one more time, we can subtract (13) from (12), then we obtain

$$\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \ldots, \tag{14}$$

where all elements having a factor of 3 or 2 (or both) are removed. It can be seen that by repeating the procedure, the terms on the right-hand side eventually disappear, then doing it infinitely for $1/p^s$, where p is prime, we get:

$$\ldots\left(1 - \frac{1}{11^s}\right)\left(1 - \frac{1}{7^s}\right)\left(1 - \frac{1}{5^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\zeta(\tag{15}$$

that is,

$$\zeta(s) = \frac{1}{\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\left(1-\frac{1}{7^s}\right)\left(1-\frac{1}{11^s}\right)\ldots}.$$

This can be written as an infinite product over all primes p , which give as result the Euler Product formula:

$$\zeta(s) = \prod_{primes} \frac{1}{1-p^{-s}}, \ R(s) > 1. \tag{16}$$

The Euler Product formula (EP) is also useful to prove that the prime numbers are infinite set as follows: since left hand side tends to infinity as $s \to 1$, then on the right-hand side the product can not have finite factors because every individual factor converges when $s \to 1$, therefore there must be infinite factors, which imply we have infinity prime numbers.

Now, we are able to take the logarithm of equation (16):

$$log\, \zeta(s) = \log log\left(\prod_p \frac{1}{1-\frac{1}{p^s}}\right) = -\sum_p \log log\left(1 - \frac{1}{p^s}\right),$$

if we apply the Taylor expansion given by $\log log\,(1 - x) = -\sum_{k=1}^{\infty} \frac{x^k}{k}$, where $x = \frac{1}{p^s}$ (Abramowitz and Stegun, 1970), then we have:

$$log\, \zeta(s) = \sum_p \left[\sum_{k=1}^{\infty} \frac{1}{kp^{ks}}\right] = \sum_{p_n} \sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{1}{p_n}\right)^{ks}, \tag{18}$$

where in the second equality, the sum has been rewritten in terms of the n-th prime p_n, Last equation demonstrates that $log\, \zeta(s)$ is a sum over all primes and their powers. Using the result in equation (18), the Prime Number Theorem can be established. In 1790, Gauss and Legendre separately conjectured the *Prime Number Theorem (**PNT**)*, that describes the asymptotic distribution of the prime numbers among the positive integers. That is, $\pi(x) \sim \frac{x}{log(x)}$ as $x \to \infty$, or alternatively $\frac{\pi(x)}{x/log(x)} \to 1$, for enough large values of x, where $\pi(x)$ is the prime counting function defined above. Figure 1 shows a numerical calculation that verify the conjecture.

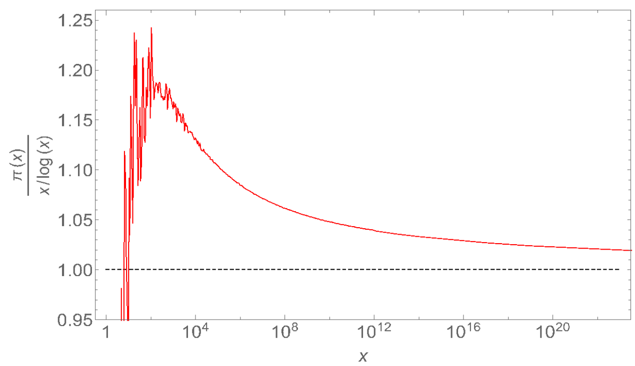


Figure 1: Behave of $\pi(x)$. It can be observed $\frac{\pi(x)}{x/loglog\,(x)}$ as $x \to \infty$.

It was much later, at the end of the 19th century, that PNT was demonstrated, independently by Jacques Hadamard (Hadamard, 1896) and Charles-Jean de la Vallée Poussin (Poussin, 1897), via a new analytic approach that was not available to Gauss and his contemporaries.

At this point, we have just considered real or complex variables whose real part is greater than 1. Riemann was the first person to introduce the whole complex plane. First of all, the Euler product formula is still legitimate, so we can write as

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_p \frac{1}{1-\frac{1}{p^s}}, \tag{19}$$

valid for $s\epsilon C$ with $R(s) > 1$. The extension to complex plane enhances the analysis in the sense that if one finds a bad point, we can avoid it by going around it. Thus, Riemann found that equation (19) has analytic continuation to whole complex plane except for the simple pole at $s = 1$. In next section we show the analytical extension of the Z function.

2.1. Analytic Continuation to real complex plane

The valid representation of $\zeta(s)$ function for $R(s) > 1$ is given by:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{20}$$

we can rewrite it for convenience as:

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^s} + \sum_{k=1}^{\infty} \frac{1}{(2k)^s} \tag{21}$$

On the other hand, the Dirichlet eta function $\eta(s)$ is defined by the following series:

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \ldots, \tag{22}$$

which is absolutely convergent for $R(s) > 1$ (Titchmarsh, E. C., and Heath-Brown, D. R. 1986). By taking the difference between (21) and (22) we have:

$$\zeta(s) - \eta(s) = 2\sum_{n=1}^{\infty} \frac{1}{(2n)^s}, \qquad = 2^{1-s}\zeta$$

Being as a valid equation in the real complex plane with a pole at $s = 1$, that is,

$$\zeta(s) = \frac{1}{1-2^{1-s}}\eta(s). \tag{23}$$

For subsequent section, the Z function will be analytically continued to the whole complex plane.

2.2. Analytic Continuation to the whole complex plane

Now, the aim is to show that $\zeta(s)$ is valid for any $s\in C$, which is achieved by finding its functional form and demonstrating that each component is valid for any s. This method is based in one of the Riemann's proofs (Titchmarsh and Heath-Brown, 1986). For a more detailed survey on the history of the functional equation see Ivić (2012).

We start with the Gamma Function, which is defined as

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt, \tag{24}$$

taking $z = \frac{s}{2}$, we have:

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} t^{\frac{s}{2}-1}e^{-t}dt. \tag{25}$$

Equation (25) is valid in the region $\{s\epsilon C: R(s) > 0\}$. Now, if we make a change of variable such that $t = \pi n^2 x$, last equation takes the form:

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} \left(\pi n^2 x\right)^{\frac{s}{2}-1} e^{-\pi n^2 x} d\left(\pi n^2 x\right). \tag{26}$$

Therefore

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\frac{1}{n^s} = \int_0^{\infty} e^{-\pi n^2 x} x^{\frac{s}{2}-1} dx,$$

summing over n,

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\sum_{n=1}^{\infty}\frac{1}{n^s} = \sum_{n=1}^{\infty}\int_0^{\infty} e^{-\pi n^2 x} x^{\frac{s}{2}-1} dx,\ \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \sum_{n=1}^{\infty}\int_0^{\infty} e^{-\pi n^2 x} x^{\frac{s}{2}}$$

interchanging the order of summation and integration operations[1]

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^{\infty}\sum_{n=1}^{\infty} e^{-\pi n^2 x} x^{\frac{s}{2}-1} dx. \tag{27}$$

Now, if we define

$$\psi(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x}, \tag{28}$$

then the equation (27) can be written as:

$$\begin{aligned}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \int_0^{\infty} x^{\frac{s}{2}-1}\psi(x)dx \\ &= \int_0^{1} x^{\frac{s}{2}-1}\psi(x)dx + \int_1^{\infty} x^{\frac{s}{2}-1}\psi(x)dx.\end{aligned} \tag{29}$$

We need to evaluate both integrals, to do it we employ the important identity given in equation (82) which comes from the Fourier analysis (see Appendix A), and from equations (28) and (29), we can operate as:

$$\sum_{n=-\infty}^{-1} e^{-\pi n^2 x} + 1 + \sum_{n=1}^{\infty} e^{-\pi n^2 x} = \frac{1}{\sqrt{x}}\left\{\sum_{n=-\infty}^{-1} e^{\frac{-\pi n^2}{x}} + 1 + \sum_{n=1}^{\infty} e^{\frac{-\pi n^2}{x}}\right\},$$

$$2\psi(x) + 1 = \frac{1}{\sqrt{x}}\left[2\psi\left(\frac{1}{x}\right) + 1\right],$$

$$\psi(x) = \frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2}, \tag{30}$$

inserting the value founded for $\psi(x)$ in equation (29), we have:

[1] Inversion can be justified by Monotone convergence, see for example (Marco, 1998).

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\int_0^1 x^{\frac{s}{2}-1}\left[\frac{1}{\sqrt{x}}\psi\left(\frac{1}{x}\right)+\frac{1}{2\sqrt{x}}-\frac{1}{2}\right]dx$$
$$+\int_1^\infty x^{\frac{s}{2}-1}\psi(x)dx$$
$$=\frac{1}{s-1}-\frac{1}{s}+\int_0^1 x^{\left(\frac{s}{2}-\frac{3}{2}\right)}\psi\left(\frac{1}{x}\right)dx$$
$$+\int_1^\infty x^{\left(\frac{s}{2}-1\right)}\psi(x)dx, \quad (31)$$

simplifying,

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\frac{1}{s(s-1)}+\int_1^\infty\left[x^{\left(-\frac{s}{2}-\frac{1}{2}\right)}+x^{\left(\frac{s}{2}-1\right)}\right]\psi(x \quad (32)$$

Since the properties of the gamma function, the left-hand side (L.H.S.) of equation (32) is valid $\forall s \in C$ with $R(s) > 1$. Now, we have to prove that the right-hand side (R.H.S.) of (32) is convergent for all $s\epsilon C$ as well. Clearly, the first term is valid except for the poles in $s = 0, 1$, then we only consider the integral term. Let us name it as I and rewrite it using definition in (28), then we have:

$$I=\int_1^\infty\sum_{n=1}^\infty x^{-\frac{s}{2}-\frac{1}{2}}e^{-\pi n^2 x}dx+\int_1^\infty\sum_{n=1}^\infty x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx. \quad (33)$$

In order to prove the convergence of I, we employ the integral test, which states: If $f(x)$ is a continuous, positive and decreasing function on the interval $[1,\infty)$, then the improper integral $\int_1^\infty f(x)dx$ converges if and only if $\sum_{n=1}^\infty f(n)$ converges. A detailed proof can be found in Apostol (2013, sec. 3.9) and Arfken et al. (2011, sec. 9.2). See also Acuña (2005) for treatment in Spanish.

In view of the integral test, we have to prove that the summations on (33) are convergent. Let us invoke the ratio test (Arfken et al., 2011), which establishes:

$$\left|\frac{a_{n+1}}{a_n}\right|<1\Rightarrow\sum_{n=1}^\infty a_n\ converge,$$
$$\left|\frac{a_{n+1}}{a_n}\right|>1\Rightarrow\sum_{n=1}^\infty a_n\ diverge.$$

So, for the first term of function I, we have:

$$\frac{e^{-(n+1)^2\pi x}x^{-\frac{s}{2}-\frac{1}{2}}}{e^{-n^2\pi x}x^{-\frac{s}{2}-\frac{1}{2}}}=\frac{e^{-(n^2+1+2n)\pi x}}{e^{-n^2\pi x}}=0<1. \quad (34)$$

This is a convergent series according to the ratio test. Analogously, for the second term of (33) we obtain a convergent series as well for any value of s. We proved that the series in equation (33) converges for $R(s) > 1$. To complete the argument, we must also verify that the corresponding integral converges.

The integrand in equation (33) consists of sums of terms of the form $f(x, s) = x^{\alpha(s)}e^{-\pi n^2 x}$. Since the exponential factor decays faster than any power of x, and

$$\sum_{n=1}^\infty e^{-\pi n^2 x}$$

remains uniformly bounded for $x \geq 1$, it follows that there exists a constant $C > 0$ such that

$$|f(x, s)|\leq Cx^{-R(s)},$$

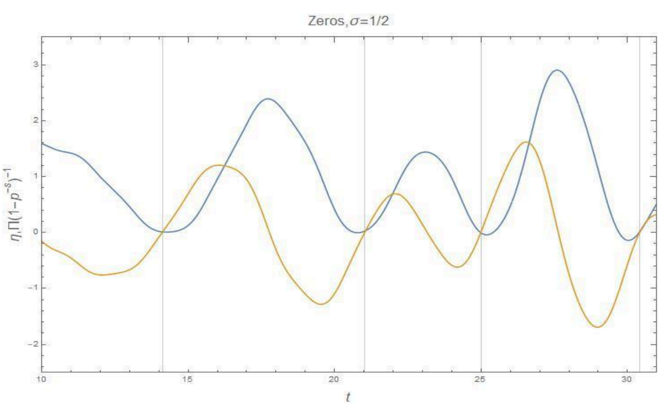
for sufficiently large x. Hence, the integrand is dominated by a power $x^{-R(s)}$ as $x\rightarrow\infty$. Then, the integrand behaves as $x^{-R(s)}$ for large x. Thus, the improper integral $\int_1^\infty f(x,s)dx$ converges whenever $R(s) > 1$, because $\int_1^\infty x^{-R(s)}dx=\frac{1}{R(s)-1}<\infty$. Therefore, both series and its integral counterpart are convergent in the same region, as required by integral test (see also Acuña, 2005; Marco, 1998). Thereby, function I is convergent for $s\epsilon C$ with $R(s) > 1$. Furthermore, we can apply the change $s\rightarrow 1 - s$ on the R.H.S of (32) and observe that the equation does not change, in fact, this change allows us finally write the functional equation, valid in the whole complex plane as:

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1}{2}+\frac{s}{2}}\Gamma\left(\frac{1}{2}-\frac{s}{2}\right)\zeta(1-s), \quad (35)$$

which is the functional equation valid for $s\epsilon C$. Although $\Gamma\left(\frac{s}{2}\right)$ has a simple pole at $s = 0$, the Riemann Zeta function itself is regular at this point, with $\zeta(0)=-\frac{1}{2}$. Hence, the only pole of $\zeta(s)$ is the simple pole at $s = 1$. (Edwards, 1974; Titchmarsh and Heath-Brown, 1986).

2.3. *Zeros of $\zeta(s)$ function for $s = \frac{1}{2} + it$*

We show a plot of the equation (23). We denote the real part as σ and imaginary part as t. By taking $\sigma = 1/2$, and expanding the sum to $n = 100$; moreover, by varying t from 1 to 30 in order to check where are its zeros. In articles as *Watkins* (2008), they report that the first zeros are at: 14.13472514, 21.02206936, 25.01085758, 30.42487612. Figure 2(d) shows first nontrivial zeros for $s = \frac{1}{2} + it$.



(d)

Figure 2: We show the real and imaginary part of η (equation (23)) to observe the first non-trivial zeros for $s = \frac{1}{2} + it$.

3. Riemann Hypothesis

So far, we have only discussed about the $\zeta(s)$ function by showing its properties and obtaining the functional equation valid on the whole complex plane. However, we have not found the values of s such that $\zeta(s) = 0$, $s\epsilon C$.

At this point we are able to rewrite the functional equation using the gamma function properties. Starting from the equation (35)

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1 - s),$$

we multiply both sides by $\frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}$ to isolate $\zeta(s)$:

$$\zeta(s) = \pi^{s-\frac{1}{2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\zeta(1 - s).$$

Now, we use the duplication formula of the Gamma function $\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$, with $z = (1 - s)/2$ (Abramowitz and Stegun, 1970), so we have

$$\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(1 - \frac{s}{2}\right) = 2^{s}\sqrt{\pi}\Gamma(1 - s),$$

in order to obtain $\Gamma\left(1 - \frac{s}{2}\right)$, we employ the reflection formula of the gamma function $\Gamma(z)\Gamma(1 - z) = \pi/\sin sin\,(\pi z)$ with $z = s/2$ (Abramowitz and Stegun, 1970), we obtain:

$$\Gamma\left(1 - \frac{s}{2}\right) = \frac{\pi}{\Gamma\left(\frac{s}{2}\right)\sin sin\left(\frac{\pi}{2}s\right)},$$

combining both results, leads to

$$\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} = 2^{s}\sin sin\left(\frac{s}{2}\pi\right)\pi^{-1/2}\Gamma(1 - s),$$

substituting last result into the equation we obtain for $\zeta(s)$, we finally have:

$$\zeta(s) = 2^{s}\pi^{s-1}\sin sin\left(\frac{\pi s}{2}\right)\Gamma(1 - s)\zeta(1 - s), \qquad (36)$$

in this form it is clear that some zeros are found at negative even integers (the positive even integer does not give a zero of the Zeta function because the zeros of the sine function are canceled by the poles of the gamma function), these are called the *trivial zeros* in the sense that they are relatively easy to find. However, we are interested on the *nontrivial zeros*, which are closely related to the prime number distribution.

It is known that the nontrivial zeros are complex numbers whose real part is in range $(0, 1)$ (Ivić, 2012). This region is called *critical strip*. Moreover, *the critical line* is $R(s) = 1/2$ named after Riemann, who ensures that those all-nontrivial zeros are contained in it, in other words, the real part of all nontrivial zeros of the Riemann Zeta function is $\frac{1}{2}$.

This statement is known as *The Riemann Hypothesis (RH)*. Several attempts have been done aimed to find nontrivial zeros with different real part. However, they have not been successful. In contrast, plenty zeros have been found (about billions of them), in fact, Godfrey Hardy proved that there are infinite zeros on the critical line (Hadamard, 1896). Bernhard Riemann was the first mathematician who got three nontrivial zeros, later Alan Turing found more than a thousand, and so on. Despite of all of this numerical evidence, those results are not enough to say that the RH is true. Opinions are actually divided in the mathematical community; some people are really confident that the RH is true while others are skeptical about it.

The goal of this work is to clarify the Guilherme França and André LeClair idea that they gave on the article: “On the validity of the Euler product inside the critical strip” (França and Le Clair, 2014). They established that the RH can be demonstrated to be true by the relation of the Zeta function and the Euler Product, in particular if the Euler Product is convergent for $R(s) < 1/2$ in a specific way (numerically) and combining this result with the functional equation. We will review some theoretical aspects, afterwards some numerical results will be shown.

1.1. *Implications of the Riemann Hypothesis*

A proof or disproof of the Riemann Hypothesis (RH) is considered the most important unsolved problem in pure mathematics. In 1900, David Hilbert published a list of twenty-three problems that were not solved at the time, and several of them were very influential for 20th-century mathematics, the RH was on that list.

Moreover, on May 24, 2000, the “Clay Mathematics Institute” stated seven problems, which are known as *Millennium Prize Problems* whose correct solution will be awarded with one million dollars prize and also the RH was among them. In fact, the RH is the only problem present in both lists, then you can imagine how important is, but why? As a short answer, this is mainly due to the relation between the RH and the distribution of the prime numbers, therefore let’s review it.

The prime counting function has been defined in the Introduction as a function that counts prime numbers less than or equal to x, but also may be written as (see Riemann, B., 1859):

$$\pi(x) = \sum_{n\geq 1}\frac{\mu(n)}{n}J\left(x^{\frac{1}{n}}\right), \qquad (37)$$

where (see Edwards, 1974; Ingham, 1990)

$$J(x) = Li(x) - \sum_{\rho}Li\left(x^{\rho}\right) + \int_{x}^{\infty}\frac{dt}{t\left(t^{2}-1\right)\log log\, t} - \log lo \qquad (38)$$

and $\mu(n)$ is the Möbius function, ρ is a zero on the critical strip; $Li(x)$ is the logarithmic integral function given by

$$Li(x) = \int_{x}^{\infty} \frac{dt}{\log log\ t}.$$

The logarithmic integral is important in a number theory because at $x \to \infty$, $\pi(x) \sim Li(x)$, which implies that we can model the development of primes, asymptotically at least. Moreover, if we assume the RH is true, we get the even stronger relation (see Titchmarsh and Heath-Brown, 1986; Ingham,1990):

$$Li(x) - \pi(x) = O\left(\sqrt{x} \log log\ x\right), \tag{39}$$

and then we would have a better distribution of the prime numbers. Otherwise, we would have a lower accuracy as the difference would have to be of $O\left(x^{\frac{1}{2}+\varepsilon}\right)$, $\forall \varepsilon > 0$. The primes are the building blocks of all numbers, if one takes any number, it can be decomposed into primes and that is why so important to understand their behavior is. Next section is aimed to address the idea of the RH is true in a numerical sense.

2. Guillerme França and André LeClaire idea

The França and André LeClair idea relied on the Riemann Hypothesis (RH) can be proved by applying only the prime numbers properties, namely its random behavior (França and LeClair, 2014). They state that if the Euler Product (EP), which is directly related to the Zeta function, is convergent for $R(s) > 1/2$ and also with the functional equation that has been proven valid for $R(s) < 1/2$ then the nontrivial zeros are constrained to lie on the critical line, that is, the RH is true. Thus, the hard work lies in proving that the Euler product converges for $R(s) > 1/2$, which is achieved using a numerical approach. To be explicit, let´s write some equations.

In order to demonstrate the EP is valid on the region $R(s) > 1/2$, we define the Dirichlet $L(s, \chi)$-function as (Apostol, 2013; Davenport, 2000):

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{s}},$$

where $\chi(n)$ is an arithmetic function, called Dirichlet Character, valid for $R(s) > 1$. Since a Dirichlet character is completely multiplicative, its L-function can also be written as a Euler product (general form for equation (16)) in the half-plane of absolute convergence (Granville, A., and Soundararajan, K., 2014), that is:

$$L(s,\ \chi) = \prod_{n=1}^{\infty} \left(1 - \frac{\chi(p_n)}{p_n^{s}}\right)^{-1},$$

therefore, from previous definitions, we are able to write:

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^{s}} = P(s, \chi) = \prod_{n=1}^{\infty} \left(1 - \frac{\chi(p_n)}{p_n^{s}}\right)^{-1}. \tag{40}$$

Note that the EP does not converge in the critical strip, however some series of (40) are still significant in this range, even if they do not converge in usual sense, their averages do; this method is known as Cesàro summation. Thereby, we are able to find a relation for the EP to be valid on the region $R(s) > 1/2$. To this aim, we take the logarithm of equation (40), then we have:

$$\begin{aligned} \log log\ L(s, \chi) &= \log log \left\{ \prod_{n=1}^{\infty} \left(1 - \frac{\chi(p_n)}{p_n^{s}}\right)^{-1} \right\} \\ &= \sum_{n=1}^{\infty} \frac{\chi(p_n)}{p_n^{s}} + \sum_{n=1}^{\infty} \frac{\chi(p_n)^2}{2p_n^{2s}} + \sum_{n=1}^{\infty} \frac{\chi(p_\cdot}{3p} \end{aligned} \tag{41}$$

The sum of second order and higher is convergent for $R(s) > 1/2$, we will show a proof of this later. Consequently, we only have to show that the first term also converges for $R(s) > 1/2$. In order to present such proof, we will work with the real part of the aforementioned term because the imaginary part has a similar behavior. The real part is given by:

$$B_N(t, \chi) = \sum_{n} \cos cos\ (t \log log\left(p_n\right) - \theta_n)\ , \tag{42}$$

where $\theta_n = -\ \log log\ (\chi(p_n))$ (França and LeClair, 2014). It is worth noting that equation (42) is not exactly the real part, since a multiplicative factor is omitted. The factor in question is $p^{-\sigma}$ (see equation (57)); it is real, positive and monotone decreasing with respect to p. Therefore, it only attenuates the amplitude of each term without modifying its oscillatory nature. In particular does not affect the order of growth of the partial sums. Consequently, it is sufficient to analyze the equation (42) in order to capture the dominant oscillatory behaviour of the leading contribution. For this reason, analysing $B_N(t, \chi)$ is sufficient to study the dominant oscillatory behavior of the leading term and its connection with the Central Limit Theorem (as will be seen later) applied to the Euler product (see Davenport, 2000; Titchmarsh and Heath-Brown, 1986). We named equation (42) as random walk primes in the sense that if this trigonometric series grows as $\sqrt{N}$ then they behave like random walks begin valid for $R(s) > 1/2$. Therefore, according (41), the $\log log\ P(s, \chi)$ will converge in $R(s) > 1/2$ as follows:

$$\log log\ L(s, \chi) = \langle \log log\ P(s, \chi) \rangle,\ valid\ for\ R(s) \tag{43}$$

where $\langle \log log\ P(s, \chi) \rangle$ denotes Cesàro average, and finally as a consequence, the Euler Product on equation (40) will be valid in this region as well.

If we can prove all previous statements, then we can conclude that there are not nontrivial zeros in $R(s) > 1/2$, otherwise we would have an infinity when we took the logarithm of zero. On the other hand, functional equation tells us there are not zeros in the region $R(s) < 1/2$ either. Thus, if we apply all requirements, we can finally conclude that the nontrivial zeros are on the critical line, which proves the RH is true.

Having presented the general idea, we can see that it is necessary to review several functions and invoke the main ingredient: The Central Limit Theorem (CLT) which is used to show the increasing $\sqrt{N}$ behavior of B_N.

1.1. Euler Product convergence in the region $R(s) > 1/2$

First of all, we must understand how to obtain the form of the equation (40), *i.e.*, the generalization of the EP, for which we previously defined the Dirichlet $L(s, \chi)$-function as:

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad s\epsilon C, \tag{44}$$

where $\chi(n): Z \to C$ is called Dirichlet Character which has the following features:

1) $\exists k\epsilon Z / \chi(n) = \chi(n + k)$;
2) Let the gcd be the grater common denominator then:
   if gcd $gcd\,(n, k) > 1 \Rightarrow \chi(n) = 0$
   if gcd $gcd\,(n, k) = 1 \Rightarrow \chi(n) \neq 0$;
3) $\chi(mn) = \chi(m)\chi(n)$;
4) From 3), $\chi(1) = 1$;
5) If $a = b(mod\ k)\ then\ \chi(a) = \chi(b)$.

The character of period 1 is called *trivial character*. A character is called *principal* if it assumes the value 1 for arguments co-prime to its modulus and otherwise is 0. A character is called *real* if it assumes real values only. Finally, a character which is not real is called complex (Olver et al., 2019).

The sign of the character χ depends on its value at -1. Specifically, χ is said to be odd if $\chi(-1) = -1$ and even if $\chi(-1) = 1$. Below we will show some examples in order to understand better the Dirichlet character functions.

Case $k = 1$

$\chi(n) = 1\ \forall n$, for this case if $\chi(n) = e^{i\theta_n} \Rightarrow \theta_n = 0$. This is the trivial character, and its series is the Riemann Zeta function.

Case $k = 2$

$$\chi(n) = \{\gcd gcd\,(n, 2) \neq 1 \Rightarrow \chi(n) = 0\ \ \gcd gcd\,(n, k \tag{45}$$

So that:

$$n = 0, \chi(0) = 0; n = 1, \chi(1) = 1;\ n = 2, \chi(2) = 0 \tag{46}$$

We can build Table 1. The first row for each modulus corresponds to the principal character, while the others are non-principal (real or complex) characters. The subscript k specifies de modulus, while the superscript j enumerates the distinct characters modulo k ($j = 1, 2, ...$).

Table 1: Values of the Dirichlet characters for different modulus.

		n					
mod k	$\chi_k^j(n)$	**1**	**2**	**3**	**4**	**5**	**6**
3	$\chi_3^1(n)$	1	0	1	0	1	0
3	$\chi_3^2(n)$	1	−1	0	1	−1	0
4	$\chi_4^1(n)$	1	0	1	0	1	0
4	$\chi_4^2(n)$	1	0	−1	0	1	0
5	$\chi_5^1(n)$	1	1	1	1	0	1
5	$\chi_5^2(n)$	1	i	$-i$	−1	0	1
5	$\chi_5^3(n)$	1	$-i$	i	−1	0	1
8	$\chi_8^1(n)$	1	0	1	0	1	0
8	$\chi_8^2(n)$	1	0	$-i$	0	−1	0
6	$\chi_6^1(n)$	1	0	0	0	1	0
6	$\chi_6^2(n)$	1	0	0	0	−1	0

Now we can assume that if $\chi(n)$ is Dirichlet character, then:

[1]. $L(s, \chi)$ has series representation with $R(s) > 0$ if χ is neither not trivial and not principal;

[2]. Since χ is completely multiplicative then equation (44) can be written as (see Poussin,1897).

$$L(s, \chi) = \prod_p \left(1 - \chi(p)p^{-s}\right)^{-1}\ \ for\ R(s) > 1; \tag{47}$$

[3]. $L(s, \chi) \neq 0$ for $R(s) > 1$; For a proof of the sentences [1] and [3], see Apostol (2013), Davenport (2000), and Titchmarsh and Heath-Brown (1986).

The main result is shown in equation (47), which we can also express as:

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = P(s, \chi) = \prod_{n=1}^{\infty} \left(1 - \frac{\chi(p_n)}{p_n^s}\right)^{-1}, \tag{48}$$

which is valid for $R(s) > 1$, however we are going to analyze that, in fact, is also valid for $R(s) > 1/2$ in a numerical sense. Firstly, we take the logarithm of equation (48):

$$\log log\ P(s, \chi) = -\sum_{n=1}^{\infty} \log log \left(1 - \frac{\chi(p_n)}{p_n^s}\right)$$
$$= \sum_{n=1}^{\infty} \left\{ \frac{\chi(p_n)}{p_n^s} + \frac{\chi(p_n)^2}{2p_n^{2s}} + \frac{\chi(p_n)^3}{3p_n^{3s}} + \ldots + \frac{\chi(p_n)^k}{kp_n^{ks}} + .. \right. \tag{49}$$

Second term and higher converge absolutely for $R(s) > 1/2$, in fact its sum is also convergent in this region.

1.2. Proof of convergence of the two-order term and higher in the region $R(s) > 1/2$

Writing a general form of the previous summations as $\sum_{n=1}^{\infty} \frac{1}{kp_n^{ks}}$, then they can be approximate to an integral in such a way that:

$$\sum_{n=1}^{\infty} \frac{1}{kp_n^{ks}} < \int_1^{\infty} \frac{dx}{kx^{ks}} = \frac{1}{k}\left[\frac{x^{-ks+1}}{-ks+1}\right]\Big|_1^{\infty}, \quad (50)$$

since $k \geq 2$, if $s = \sigma + it$, with $\sigma \geq \frac{1}{2}$, then $ks - 1 = k\sigma - 1 + ikt$ where $k\sigma - 1 > 0$. Thus $\frac{1}{-ks+1}$ is well defined for $R(s) > 1/2$, and the value of the integral goes to zero when $x \to \infty$, then we have:

$$\int_1^{\infty} \frac{dx}{kx^{ks}} = \frac{1}{k(ks-1)}. \quad (51)$$

The summation $\sum_{k=2}^{\infty} \frac{1}{k(ks-1)}$ is convergent, and for large k it will behave as:

$$\sum_{k=2}^{\infty} \frac{1}{k(ks-1)} \sim \sum_{k=2}^{\infty} \frac{1}{k^2 s} \sim \frac{1}{s} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{s}\frac{\pi^2}{6}, \quad (52)$$

The argument of sum can be factorized as $k(ks-1) = k^2 s(1 - \frac{1}{ks})$ for an infinity sum, the second term tends to zero, which implies that approximation is correct to infinite sum, which proves that the sum is convergent. Therefore, the logarithm is:

$$\log log\, \zeta(s) < \sum_{n=1}^{\infty} \frac{1}{p_n^s} + \sum_{k=2}^{\infty} \frac{1}{k(ks+1)}, \quad (53)$$

Then, the final expression is

$$\log log\, \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{p_n^s} + O(1),\ for\ R(s) > 1/2. \quad (54)$$

At this point, we can conclude that the whole convergence of equation (49) depends on the first term, therefore if we can prove that term does converge in $R(s) > 1/2$ then the EP does as well. We will do it by working with some trigonometric functions, as follows.

1.3. *Proof of convergence of the one order term in the region* $R(s) > 1/2$.

Rewriting the first term of right-hand side of equation (49) and naming it as $X(s, \chi)$ such that:

$$X(s, \chi) = \sum_{n=1}^{N} \frac{\chi(p_n)}{(p_n)^s}, \quad (55)$$

we are able to take its real part named it as follows:

$$S_N(s, \chi) = R\{X_N(s, \chi)\}. \quad (56)$$

We derive its explicit form. We know that $\chi(p_n) = e^{i\theta_n}$ and $s = \sigma + it$, then:

$$\frac{e^{i\theta_n}}{p_n^{\sigma+it}} = \frac{1}{p_n^{\sigma}}\left[\cos cos\left(\theta_n - t \log log\, p_n\right) + i \sin sin\left(\theta_n - t \log log\, p_n\right)\right].$$

Afterwards, the equation (56) becomes as

$$S_N(s, \chi) = \sum_{n=1}^{N} \frac{1}{p_n^{\sigma}} \cos cos\left(\lambda_n\right), \quad (57)$$

where $\lambda_n = t \log log\, p_n - \theta_n$ and $\theta_n = - i \log log\, \chi(p_n)$. In order to work with trigonometric series, let´s set

$$C(N) = \sum_{n=1}^{N} \cos cos\,(\lambda_n). \quad (58)$$

If we prove $C(N) \propto O(\sqrt{N})$, then we will be able to demonstrate that $X(s, \chi)$ is a convergent series. First, we have to prove the $\sqrt{N}$ statement, once done it, we will prove the convergence of the summation in equation (55).

Expanding the argument of equation (58) as follows:

$$\begin{aligned} \cos cos\left(\lambda_n\right) &= \sum_{k=1}^{n} \cos cos\left(\lambda_k\right) - \sum_{k=1}^{n-1} \cos cos\left(\lambda_k\right) \\ &= C(n) - C(n-1),\ \text{for}\ n \geq 2, \end{aligned} \quad (59)$$

plug (59) in (57) we have:

$$\begin{aligned} S_N(s, \chi) &= \sum_{n=2}^{N} \frac{C(p_n) - C(p_{n-1})}{p_n^{\sigma}} \\ &= \sum_{n=1}^{N-1} C(p_n)\left[\frac{1}{p_n^{\sigma}} - \frac{1}{(p_{n+1}^{\sigma})}\right] + \frac{C(p_N)}{p_N^{\sigma}}, \end{aligned}$$

where $\sigma = R(s)$, therefore

$$\begin{aligned} S(s, \chi) &= \sum_{n=1}^{\infty} C(p_n)\left[\frac{1}{p_n^{\sigma}} - \frac{1}{(p_{n+1}^{\sigma})}\right] \\ &= \sigma \sum_{n=1}^{\infty} C(p_n) \int_{p_n}^{p_{n+1}} \frac{du}{u^{\sigma+1}} \\ &= \sigma C(p_1)\int_2^{p_2} \frac{du}{u^{\sigma+1}} + \sigma C(p_2)\int_{p_2}^{p_3} \frac{du}{u^{\sigma+1}} + \sigma C(p_3)\int_{p_3}^{p_4} \frac{du}{u^{\sigma+1}} + \ldots, \\ &= \sigma\left\{\int_2^{p_2} \frac{C(u)du}{u^{\sigma+1}} + \int_{p_2}^{p_3} \frac{C(u)du}{u^{\sigma+1}} + \ldots\right\}, \\ &= \sigma \int_2^{\infty} \frac{C(u)du}{u^{\sigma+1}}, \end{aligned} \quad (60)$$

where $C(u) = C(p_n)$ for $p_n < u < p_{n+1}$. To understand the conditions for this integral to have a convergent value, we apply the result:

$$\int_{2}^{\infty} \frac{dx}{x^{\alpha}} = \{\frac{2^{1-\alpha}}{\alpha-1},\ \alpha\geq 1,\ \infty,\ \alpha < 1, \tag{61}$$

then, we are able to write

$$S(s,\chi) = \sigma C(u)\int_{2}^{\infty} \frac{du}{u^{\sigma+1-\beta}}\ where\ C(u) \sim u^{\beta}, \tag{62}$$

for which the usual ansatz of modeling the growth of partial sums as u^{β} has been employed (Iwaniec, and Kowalski, 2004).
For equation (62) to be convergent for $\sigma > 1/2$, we take the result (61) and observe that $\beta \leq 1/2$, then we can conclude that C_N must grow as $\sqrt{N}$.

We have to prove now the convergence of $X(s,\chi)$, to achieve it, let us recall another important subject: Central Limit Theorem (Grinstead, 1997). We will apply it to prove that if certain trigonometric series behaves as independent and identically distributed random variables then they satisfy the CLT thereby they will be a convergent series. Consider the series:

$$C_N(u) = \sum_{n=1}^{N} \cos cos\left(u\lambda_n\right), \tag{63}$$

if $u = 1$ we have same series as in equation (58), and u is a uniformly distributed real variable on the interval $[0, 2\pi]$. The CLT applies to the series (63) if the $\lambda_n^{'}s$ are linearly independent over the integers, it is easy to prove that our λ_n have this property, we are no going to prove it but you can see Borwein et al. (2008), for instance. Thus, since the $\lambda's$ in (57) are linearly independent we can define $B_N(u; t, \chi)$ as follows:

$$B_N(u; t, \chi) = \sum_{n=1}^{N} \cos cos\left[u\left(t \log log\, p_n - \theta_n\right)\right], \tag{64}$$

which satisfies the CLT, then we are able to guarantee that the $B_N/\sqrt{N}$ is finite for any $u\neq 0$, as a consequence, the EP through $L(s,\chi)$ will converge for $R(s) > 1/2$. Now, there is a very important difference between principal and non-principal characters, let us show what are the implications of work with each of them.

Furthermore, given that $C(u) = \sum_{p_n\leq u} a_n$, $a_n = \cos cos\left(t \log log\, p_n\right)$, each oscillatory term a_n is independent and bounded, under the assumption that the values of $\log log\, p_n$ act as pseudorandom seeds (see LeClaire's article), the a_n can be modeled as oscillatory variables with mean value is zero, and variance is of $O(1)$. According to the CLT argument, the sum of N of these terms has a standard deviation of $O(\sqrt{N})$. Since $N = \pi(u) \sim \frac{u}{\log log\, u}$, we obtain

$$C(u) = O\left(\sqrt{\frac{u}{\log log\, u}}\right) = u^{1/2}(\log log\, u)^{-1/2},$$

that is, leading exponent of $\beta = \frac{1}{2}$ (for simplicity, logarithm factors are dropped in the notation u^{β}).

Let $\Delta\phi_n$ such that:

$$\Delta\phi_n = \lambda_{n+1} - \lambda_n \Delta\phi_n = t \log log\, p_{n+1} - t \log l \tag{65}$$

where $g_n = p_{n+1} - p_n$, now since $g_n/p_n \to 0$ due to $g_n \sim \log log\, n$, then we have to impose $\Delta\theta_n \neq 0$ which is true when χ is not principal and therefore $C_N \sim \sqrt{N}$. However, if $\chi = 1$ is principal this implies that $\lambda_{n+1} - \lambda_n \approx t g_n/p_n$. Moreover, if $t = 0$ then $C_N \sim N$, as $n\to\infty$ and we can not apply the CLT so the sum in (63) is divergent. To avoid this unwanted behavior, a solution was proposed by Guilherme França and André LeClair. They found that if t is large enough and $N < N_c$ where $N_c \propto t^2$ then $C_N \propto \sqrt{N}$ for the principal Dirichlet character as well. The parameter N_C is the critical cutoff separating the random walk regime from the correlated regime of the partial sums (França & LeClair).

Let us summarize the statement: There is a problem in applying CLT to the series B_N when the Dirichlet character is principal, however it is possible to still have $B_N = O\left(\sqrt{N}\right)$ valid below a cut-off given by N_c, this value depends on $N_c \propto t^2$ (see França and LeClair, (2014), pp.12 for a proof of this quadratic behavior). This is the most important contribution made by Guilherme França and André LeClair in their article, therefore we want to present a numerical verification in the next section.

2. Numerical Verification

We show numerically test of the previous conclusions. First of all, for principal Dirichlet character, equation (42) becomes:

$$B_N(t) = \sum_{n=1}^{N} \cos cos\left(t \log log\, p_n\right). \tag{66}$$

We want to check that function (66) is bounded by $\sqrt{N}$ (same is true for non-principal Dirichlet L-function through (64). In Figure 3 we plot the partial sums $\left|B_N\right|$ and compare them with $\sqrt{N}$. The figure shows that $\left|B_N\right|$ remains below $\sqrt{N}$, indicating that its growth is at most of order $\sqrt{N}$.

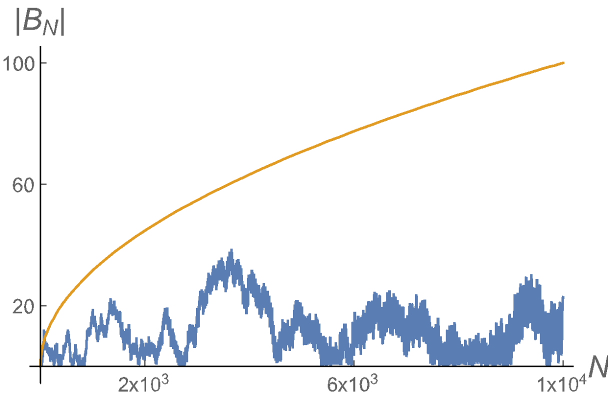
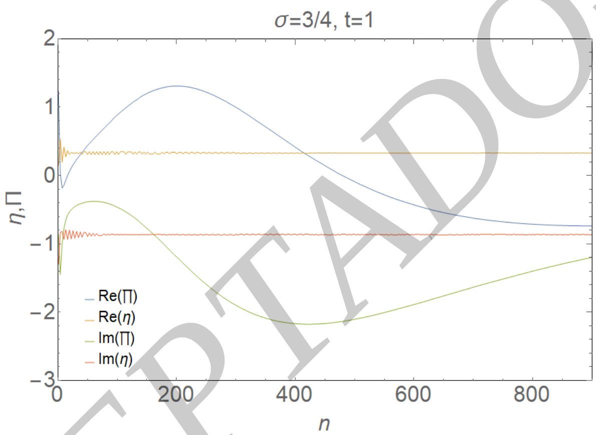
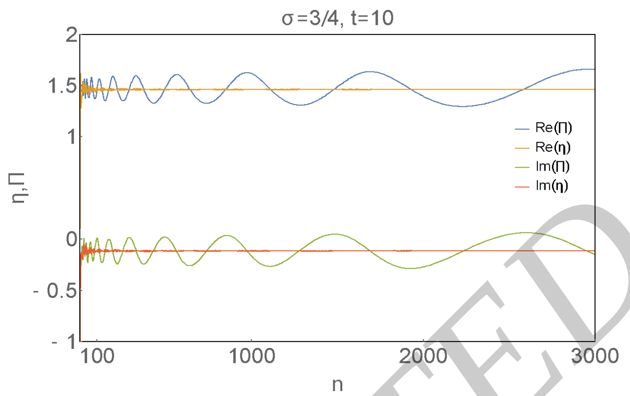


Figure 3: The absolute value of the partial sum B_N versus N for a fixed $t = 10^3$ (blue line); function $\sqrt{N}$ (yellow line).

Further, in Figure 4 we show two more plots in order to compare the behaviors of the Z function defined in equations as the Euler Product in (19) and as in equation (23), where $s \epsilon C$ such that $s = \sigma + it$, we denote the real part as σ and imaginary part as t; n is the sum index, for different values of t. The first one, is perform by taking $\sigma = 3/4$, $t = 1$ and varying n from 0 to 1000. We can observe the behavior is clearly different since the beginning. For second plot, we have chosen $\sigma = 3/4$, $t = 10$, and varying n from 0 to 3000. We can observe that EP and Zeta function calculated by using equation (23) almost match at $N_c \propto t^2 = 100$ then $N_c \propto 100$, after that point they goes separately as we expected.



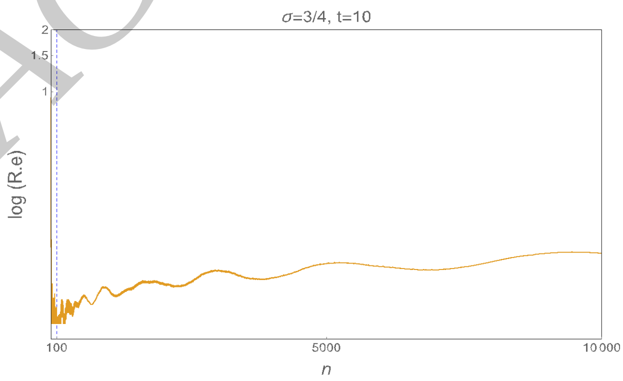
(a)



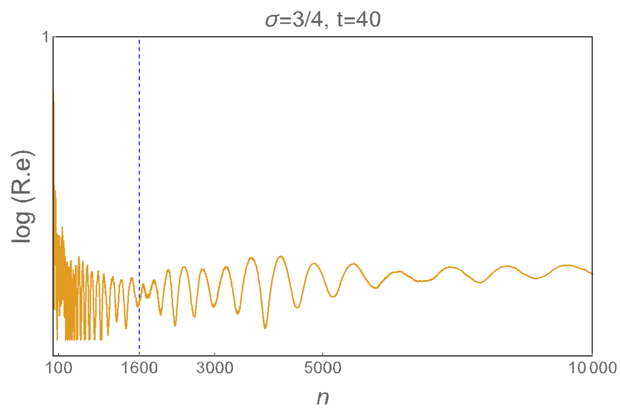
(b)

Figure 4: Behavior of the Zeta function defined through the Euler product and the $\eta(s)$ function. Plots show the $\zeta(s)$ and $\eta(s)$ functions with: (a) $\sigma = 3/4, t = 1, n = 1000$; (b) $\sigma = 3/4, t = 10, n = 3000$.

Finally, we show the relative error between the real value of the Zeta function defined in equation (23), and as the Euler Product for the region $R(s) > 1$. Let us name to the relative error as "R.e". In Figure 5 these relative errors are plot by considering $\sigma = 3/4$ and $t = 10$.



(a)



(b)

Figure 5. The relative errors between two definitions of the Zeta functions by applying the Le Claire idea. We fix $n = 10000$ for both cases. *(a)* $t = 10$ implies $N_C \propto 100$ (dashed blue line); *(b)* $\sigma = 3/4$ and $t = 40$.

3. Conclusions

As has been shown in this work, the Riemann Hypothesis is of utmost importance in number theory since, if can be proved to be true, the main consequence would be the knowledge of the distribution of prime numbers, which are the blocks with all natural numbers are built. On the other hand, we have concluded the hypothesis is true, at least numerically, however and despite of all the work that has been done so far, we have to continue investigating functions such as the Dirichlet function and its special case, the Riemann Z Function, in this sense other important results have been developed.

Agradecimientos

The work of Wendy González Olivares is supported by “Estancias Posdoctorales por México (SECIHTI)” and “Sistema Nacional de Investigadores e Investigadoras” (SNII-SECIHTI). O.F.B. and E.B.G. acknowledges support from the SECIHTI project No. CBF-2025-G-1187. O.F.B. and E.B.G. thank the support of SNII-SECIHTI, VIEP-BUAP and PRODEP, México.

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Appendix A. Fourier analysis

We need to evaluate the integrals on equation (29), to do it we have to prove an important identity which comes from the Fourier analysis, the identity is written as:

$$\sum_{n=-\infty}^{\infty} e^{-n^2\pi x} = \frac{1}{\sqrt{x}} \sum_{n=-\infty}^{\infty} e^{-\frac{k^2\pi}{x}}. \tag{69}$$

To prove it, firstly we must prove the Poisson Summation Formula (PSF), which is often written in the following form:

$$\sum_{n=-\infty}^{\infty} f(t + nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} g\left(\frac{k}{T}\right)e^{2\pi ikt/T}, \quad (70)$$

where $g\left(\frac{k}{T}\right)$ is the Fourier transform of $f(t)$ such that:

$$g\left(\frac{k}{T}\right) = \int_{-\infty}^{\infty} d\tau e^{-\frac{2\pi ik\tau}{T}} f(\tau). \quad (71)$$

(I) Proof of PSF

Assuming that $F(T)$ is a periodic function of period T, then,

$$F(t) = \sum_{n=-\infty}^{\infty} f(t + nT), \quad (72)$$

since $F(T)$ is a periodic function, we write it as a Fourier series:

$$F(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} G(k)e^{2\pi ikt/T}, \quad (73)$$

where the Fourier amplitude is given by

$$G(k) = \int_{-T/2}^{T/2} dte^{-2\pi ikt/T}F(t) = \sum_{n=-\infty}^{\infty}\int_{-T/2}^{T/2} dte^{-2\pi ikt/T} f \quad (74)$$

Now, we make the following change of variable $\tau \rightarrow t + nT$ then the limits of the integral will be $\frac{1}{2}(2n\pm 1)T$. Therefore, we write $G(k)$ as:

$$G(k) = \sum_{n=-\infty}^{\infty}\int_{\frac{1}{2}(2n-1)T}^{\frac{1}{2}(2n+1)T} d\tau e^{-\frac{2\pi ikt}{T}} e^{2\pi ikn} f(\tau)$$

$$= \sum_{n=-\infty}^{\infty}\int_{\frac{1}{2}(2n-1)T}^{\frac{1}{2}(2n+1)T} d\tau e^{-\frac{2\pi ikt}{T}} f(\tau)$$

$$= \sum_{n=-N}^{N}\int_{\frac{1}{2}(2n-1)T}^{\frac{1}{2}(2n+1)T} d\tau e^{-\frac{2\pi ikt}{T}} f(\tau)$$

$$= \left\{\int_{\frac{1}{2}(2(-N)-1)T}^{\frac{1}{2}(2(-N)+1)T} h(\tau) + \int_{\frac{1}{2}(2(-N+1)-1)T}^{\frac{1}{2}(2(-N+1)+1)T} h(\tau) + \ldots + \int_{\frac{1}{2}(2N-1)T}^{\frac{1}{2}(2N+1)T} h(\tau)\right\} wh$$

$$\int_{-\frac{1}{2}(2N-1)T}^{\frac{1}{2}(2N+1)T} d\tau e^{-2\pi ikt/T} f(\tau)$$

$$= \int_{-\infty}^{\infty} d\tau e^{-\frac{2\pi ik\tau}{T}} f(\tau),$$

$$\Rightarrow G(k) = g\left(\frac{k}{T}\right). \quad (75)$$

Therefore, $F(t)$ reads as

$$F(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G(k)e^{2\pi ikt/T}, \qquad = \frac{1}{T}\sum_{k=-\infty}^{\infty} g\left(\frac{k}{T}\right)e^{2\pi ikt/T}$$

and the Poisson summation formula is then:

$$\sum_{n=-\infty}^{\infty} f(t + nT) = \frac{1}{T}\sum_{k=-\infty}^{\infty} g\left(\frac{k}{T}\right)e^{2\pi ikt/T}, \; q.e.d., \quad (76)$$

we have proven the PSF. We are in position to prove the identity (69).

(II) Proof of equation (69)

Taking $f(t) = e^{-t^2}$ and $T = \sqrt{\pi Z}$, then we write:

$$F(0) = \sum_{n=-\infty}^{\infty} f(nT) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi z}, \quad (77)$$

by using (76), we have:

$$\sum_{n=-\infty}^{\infty} e^{-n^2\pi z} = \frac{1}{\sqrt{\pi z}}\sum_{k=-\infty}^{\infty} g\left(\frac{k}{\sqrt{\pi z}}\right), \quad (78)$$

the Fourier transform is given by

$$g\left(\frac{k}{T}\right) = \int_{-\infty}^{\infty} e^{-2\pi ikt/T} f(t)dt. \quad (79)$$

Then

$$g\left(\frac{k}{\sqrt{\pi z}}\right) = \int_{-\infty}^{\infty} e^{-2\pi ikt/\sqrt{\pi z}} e^{-t^2} dt \qquad = \int_{-\infty}^{\infty} e^{-2\sqrt{\pi}ikt/\sqrt{z}} e^{-t^2} dt$$

$$\Rightarrow g\left(\frac{k}{\sqrt{\pi z}}\right) = \sqrt{\pi}e^{-k^2\pi/z}. \quad (80)$$

Therefore, we write again,

$$\sum_{n=-\infty}^{\infty} e^{-n^2\pi z} = \frac{1}{\sqrt{\pi z}}\sum_{k=-\infty}^{\infty} g\left(\frac{k}{\sqrt{\pi z}}\right) \quad (81)$$

and hence

$$\sum_{n=-\infty}^{\infty} e^{-n^2\pi z} = \frac{1}{\sqrt{z}}\sum_{k=-\infty}^{\infty} e^{-k^2\pi/z} \quad q.e.d. \quad (82)$$