

## On the relation between the Fiedler value and the Kirchhoff index of weighted graphs Sobre la relación entre valor Fiedler y el índice de Kirchhoff de gráficas con peso

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### Resumen

Se consideran dos cantidades asociadas a una gráfica  $\Gamma$  que, en diferentes formas, miden qué tan conectada está la gráfica: la conectividad algebraica (valor Fiedler) y la resistencia efectiva total (índice de Kirchhoff). Se plantean los siguientes problemas de optimización:

1. De entre las asignaciones de pesos en las aristas de  $\Gamma$  para las que la gráfica tiene resistencia efectiva total igual a una constante dada, determinar los que maximizan la conectividad algebraica.
2. De entre las asignaciones de pesos en las aristas de  $\Gamma$  para las que la gráfica tiene conectividad algebraica igual a una constante dada, determinar los que minimizan la resistencia efectiva total.

Para las gráficas completas, mostramos que la solución al primer problema se alcanza cuando el peso en las aristas es constante. El resultado principal de este trabajo es la prueba de que el segundo problema no tiene solución cuando  $\Gamma$  es un árbol: el índice de Kirchhoff no alcanza un valor mínimo bajo la restricción de que el valor Fiedler esté fijo. Como consecuencia de esto, la razón  $\lambda_3/\lambda_2$  para valores propios del laplaciano puede hacerse arbitrariamente grande eligiendo pesos apropiados en cualquier árbol dado.

**Palabras Clave:** Conectividad de redes, gráfico Laplaciano, valor Fiedler, resistencia efectiva, índice de Kirchhoff, optimización de eigenvalores.

### Abstract

This work concerns two quantities associated with a graph  $\Gamma$  that measure, in different ways, how strongly connected the graph is: the algebraic connectivity (Fiedler value) and the total effective resistance (the Kirchhoff index). We consider the following optimization problems:

1. Among all edge weights for which  $\Gamma$  has total effective resistance equal to a prescribed constant, determine those that maximize the algebraic connectivity.
2. Among all edge weights for which  $\Gamma$  has algebraic connectivity equal to a prescribed constant, determine those that minimize the total effective resistance.

For the complete graph with  $n$  vertices, we show that the first problem is solved when all edge weights are equal. The main result of this work is that the second problem has no solution whenever  $\Gamma$  is a tree: the Kirchhoff index does not attain a minimum under the constraint of a fixed Fiedler value. As a consequence, the ratio  $\lambda_3/\lambda_2$  of Laplacian eigenvalues can be made arbitrarily large among weighted versions of any given tree.

**Keywords:** Network connectivity, graph Laplacian, Fiedler value, effective resistance, Kirchhoff index, eigenvalue optimization.

### 1. Introduction

In 1973, M. Fiedler introduced the term *algebraic connectivity* for the second-smallest eigenvalue of the graph Laplacian (Fiedler, 1973), proposing it as a quantity that describes how well connected or robust a graph is. Later, he extended the def-

inition to include graphs with weights on the edges (Fiedler, 1990). Since that starting point, the algebraic connectivity — also known as the Fiedler value — has been extensively studied by many authors (see the surveys Abreu, 2007; Mohar, 1992 and the books Chung, 2007; Cvetkovic et al., 2009).

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The total effective resistance of a graph, also known as the *Kirchhoff index*, was proposed in (Klein and Randić, 1993) as an alternative measure of connectedness for weighted graphs. This index is based on the effective resistance metric, which had previously been considered from different viewpoints (cf. Powers, 1976; Doyle and Snell, 1984; Kigami, 1994). Its historical origin dates back to the nineteenth century with Kirchhoff's work on electric circuits (Kirchhoff, 1847). A very relevant result, from the point of view of spectral graph theory, was proved in (Klein and Randić, 1993) where it was shown that the Kirchhoff index can be expressed in terms of the graph Laplacian eigenvalues (theorem 4.5).

In this work, we study optimization problems involving these two quantities. More precisely, we ask for the optimal value of one quantity when the other one is fixed. On one hand, we seek the weights on the edges that maximize the Fiedler value when the total effective resistance is prescribed. On the other hand, we consider the reciprocal question, that is to determine the minimal total effective resistance when the Fiedler value is held fixed. There is a substantial literature on optimization problems for the algebraic connectivity (Fiedler, 1990; Fallat and Kirkland, 1998; de Lima et al., 2007; Büyükoğlu and Leydold, 2009; Rojo et al., 2010), whereas minimization problems for the total effective resistance have also been studied, both from a graph-theoretic perspective and in applications (e.g. Ghosh et al., 2008; Ellens et al., 2011; Yang et al., 2019; Predari et al., 2023; Zhou et al., 2025).

## 2. Preliminaries: the graph Laplacian

A graph  $\Gamma$  consists of a set of vertices and a set of edges. An edge is an unordered pair of different vertices. In this work, we will consider both sets to be finite. A graph is usually visualized by collections of dots and lines joining them; the dots represent the vertices, and the lines represent the edges. A line between vertices  $v_i$  and  $v_j$  is drawn whenever the pair  $\{v_i, v_j\}$  belongs to the set of edges of the graph. The edge  $\{u, v\}$  is *incident* to the vertices  $u$  and  $v$ , and those vertices are *adjacent* to each other. We will also use the notation  $u \sim v$  to indicate that vertices  $u$  and  $v$  are adjacent. A graph is *connected* if for every pair of vertices  $(u, w)$  there exists a sequence of vertices  $\{v_1, v_2, \dots, v_k\}$  such that each  $v_i$  is a vertex of the graph, each pair  $\{v_i, v_{i+1}\}$  is an edge,  $u = v_1$  and  $w = v_k$ . Such a sequence is called a *path* joining  $u$  and  $w$ , if all vertices in it are distinct; it is called a *cycle* if all the vertices are distinct except for  $u = w$ . The graph that consists of only one path with  $n$  vertices is denoted by  $P_n$ , and the cycle with  $n$  vertices by  $C_n$ . It is said that  $\Gamma$  is a *tree* if it contains no cycles. Equivalently, a graph is a tree if it is connected and has the property that for each pair of different vertices there exists a unique path that joins them. The *complete graph* of  $n$  vertices, denoted by  $K_n$ , is such that each vertex shares an edge with every other vertex. The *star graph*  $S_n$  is a tree with  $n$  vertices, one of degree  $n - 1$  (the *central vertex*) and the rest with degree 1 each. The *lollipop graph*  $L_{m,n}$  is the connected graph with  $m + n - 1$  vertices, formed by a complete graph  $K_n$  and a path  $P_m$  joined together, so that one of the vertices of  $K_n$  is one of the extremal vertices of the path.

A *weighted graph* is a pair  $(\Gamma, \omega)$ , where the weight  $\omega$  is a real valued function defined on the set of edges. We will consider only graphs with positive weights and use the notation  $\Gamma_\omega$  to denote the graph  $\Gamma$  with weight  $\omega$ . The *edge weight* at  $\{u, v\}$  is the value of  $\omega$  at the edge and will be denoted by  $\omega_{u,v}$ . The case when every edge weight is equal to 1 corresponds to the *unweighted graph*  $\Gamma$ . The *degree* of a vertex  $v$  is

$$\deg(v) = \sum_{u \sim v} \omega_{u,v}.$$

In particular, for unweighted graphs the weight of a vertex is just the number of edges incident to it.

Let  $\Gamma_\omega$  be a weighted graph, with set of vertices  $V$  and set of edges  $E$ . Denote by  $\ell(V)$  the linear space of real valued functions with domain  $V$  equipped with the (real) inner product

$$\langle x, y \rangle = \sum_{v \in V} x(v)y(v). \quad (1)$$

The *Laplace operator* associated with  $\Gamma_\omega$  (or *Laplacian*) of the weighted graph is the operator  $L_{\Gamma_\omega}$  acting on  $\ell(V)$  given by

$$(L_{\Gamma_\omega} x)(u) = \sum_{\{v,u\} \in E} \omega_{u,v}(x(u) - x(v)),$$

for  $x \in \ell(V)$  and  $u \in V$ . The identity

$$\langle L_{\Gamma_\omega} x, x \rangle = \sum_{\{u,v\} \in E} \omega_{u,v}(x(u) - x(v))^2. \quad (2)$$

for the quadratic form of the Laplace operator is well known and widely used in the study of the Laplacian (e.g. Strichartz, 2006; Abreu, 2007). As a direct consequence of (2), we can see that  $L_{\Gamma_\omega}$  is a non-negative self-adjoint operator. We order the eigenvalues of the Laplacian in non-decreasing order  $0 \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ . Since  $L_{\Gamma_\omega}$  vanishes for constant vectors in  $\ell(V)$ , it is always the case that  $\lambda_1 = 0$ . Given an arbitrary order of the vertices  $V = \{v_1, v_2, \dots, v_n\}$ , consider the orthonormal basis of  $\ell(V)$  given by  $\{e_1, e_2, \dots, e_n\}$  where  $e_i(v_j) = \delta_{i,j}$ . It is easy to see that the representation of  $L_{\Gamma_\omega}$  with respect to that basis is such that the entries in the main diagonal are the degrees of the vertices, and for  $i \neq j$ , the entry in the position  $(i, j)$  is equal to  $-\omega_{v_i, v_j}$ . In particular the sum of the entries at each row and at each column is zero.

One example that is both very simple and important, is that of the complete (unweighted) graph  $K_n$ . In this case, the eigenvalues are  $\lambda_2 = \lambda_3 = \dots = \lambda_{n-1} = n$ . Indeed, it is straightforward to verify that for any pair of distinct vertices  $(u, w)$  the function

$$\varphi(v) = \begin{cases} 1, & v = u, \\ -1, & v = w, \\ 0, & \text{elsewhere,} \end{cases}$$

is an eigenvector of  $L_{K_n}$  with eigenvalue  $n$ . Since there are  $n - 1$  linearly independent such functions, the assertion is true.

### 3. Algebraic connectivity

**Definition 3.1. (Algebraic connectivity)** The *algebraic connectivity* or *Fiedler value* of a weighted graph  $\Gamma_\omega$  is the second smallest eigenvalue of the Laplacian  $L_{\Gamma_\omega}$ . The associated eigenvectors are the *Fiedler vectors* of the weighted graph.

Thus, the algebraic connectivity is the value  $\lambda_2$  associated with the weighted graph. We will write  $\lambda_2(\Gamma_\omega)$  whenever it may be necessary to specify the graph. A basic result with regard to the Fiedler value is that it equals zero if and only if the graph is disconnected. This can be easily seen from the equality (2): The right hand side of the equation is strictly positive if and only if there exists an edge  $\{u, v\}$  for which  $x(u) \neq x(v)$ . For connected graphs, such an edge will exist whenever the vector  $x$  is non-constant. If the graph is disconnected, then  $x$  can have different constant values in each connected component and  $L_{\Gamma_\omega}x$  would still vanish.

The well known Courant–Weyl–Fischer theorem (Courant and Hilbert, 1937), gives a convenient expression for the Fiedler value

$$\lambda_2 = \min_x \frac{\langle L_{\Gamma_\omega} x, x \rangle}{\langle x, x \rangle}, \quad x \neq 0, \quad \sum_{v \in V} x(v) = 0. \quad (3)$$

The right hand side of the first equality in (3) is known as the *Rayleigh quotient* of the vector  $x$ . In view of relation (2) we can write this also in the form

$$\lambda_2 = \min_x \sum_{\{u,v\} \in E} \omega_{u,v} (x(u) - x(v))^2, \\ \|x\| = 1, \quad \sum_{v \in V} x(v) = 0.$$

This implies that  $\lambda_2$  will not decrease whenever an edge is added to the graph, or if the value of a given edge weight  $\omega_{u,v}$  increases. However, there are situations for which the algebraic connectivity is not sensible enough to detect some edges: the smallest instance of this is the pair of graphs with four vertices that are shown in figure 1.

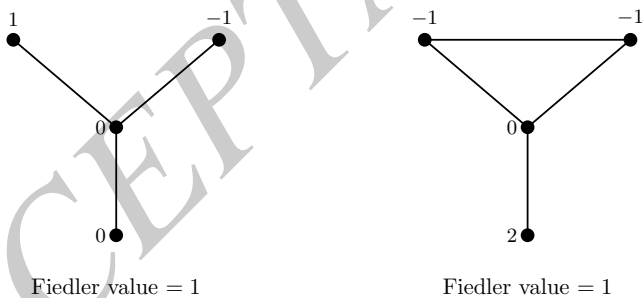


Figure 1: Graphs with the same algebraic connectivity. The corresponding Fiedler vectors are shown.

It should be noted that this is the only such example of graphs with four vertices. There are several more examples for graphs with more vertices, where it is even possible to add two edges without increasing the Fiedler value. In fact, the example in figure 1 is just the simplest case of a more general situation: every star graph  $S_n$  and every lollipop graph  $L_{1,n}$  have algebraic

connectivity equal to 1, even though the lollipop graphs have many more edges connecting the vertices.

Despite of what might be suggested by the comparison between the star and lollipop graphs, the Fiedler value is still a very important and relevant measure of the connectedness or robustness of the graph. One way to argue in favor of this, is to look at how the Fiedler value compares to different quantities related to the connectedness of the graph. Two relevant examples of this are the *edge connectivity*  $e(\Gamma)$  and the *cut function*  $\phi(\Gamma)$ . The edge connectivity represents the minimal possible value for which  $\Gamma$  can be disconnected when a set of  $e(\Gamma)$  edges is removed from the graph. Simple examples are  $e(T) = 1$  for any tree  $T$  and  $e(C) = 2$  for any cycle  $C$ . With respect to the edge connectivity, the Fiedler value satisfies the inequalities (Fiedler, 1989)

$$2e(\Gamma)(1 - \cos \frac{\pi}{n}) \leq \lambda_2(\Gamma) \leq e(\Gamma),$$

for any (unweighted) non-complete graph  $\Gamma$  with  $n \geq 3$  vertices.

The cut function  $\phi(\Gamma)$  (also known as the *isoperimetric number* and as the *Cheeger constant*) is defined by

$$\phi(\Gamma) = \min \left\{ \frac{|\partial S|}{|S|} \mid |S| \leq |V \setminus S| \right\},$$

where  $\partial S$  is the *edge boundary*, defined as the set of edges  $\{u, v\}$  for which one of the two vertices belongs to  $S$  and the other belongs to its complement  $V \setminus S$ . Thus, the cut function  $\phi(\Gamma)$  will be small whenever it is possible to separate the graph  $\Gamma$  in two connected components, both with a large number of vertices, by removing a small number of edges. The cut function provides the following lower and upper bounds for the Fiedler value:

$$\frac{\phi(\Gamma)^2}{2\Delta} \leq \lambda_2(\Gamma) \leq 2\phi(\Gamma),$$

where  $\Delta$  denotes the maximal degree of the vertices. Those relations are sometimes called the *discrete Cheeger inequalities* and we refer to (Alon and Milman, 1985; Chung, 2007; Spielman, 2019) for extensive discussions on this topic.

The *absolute algebraic connectivity* of a graph  $\Gamma$ , introduced in Fiedler, 1990, is defined by

$$\hat{\alpha}(\Gamma) = \max_\omega \left\{ \lambda_2(\Gamma_\omega) \mid \sum_{\{u,v\} \in E} \omega_{u,v} = |E| \right\}.$$

So, the absolute algebraic connectivity is the maximal possible value of the algebraic connectivity subject to the condition that the sum of the weights of the graph remains constant. Note that the sum of the weights of the graph equals one half of the sum of the degrees of the vertices; since this is the trace of the Laplace operator it equals the sum of the eigenvalues. This means that the total sum of the weights on the edges of a graph is a spectral invariant: It is determined by the eigenvalues of the Laplacian. From this discussion, we can see that the absolute algebraic connectivity can be written as

$$\hat{\alpha}(\Gamma) = \max_\omega \{ \lambda_2(\Gamma_\omega) \mid \lambda_2(\Gamma_\omega) + \dots + \lambda_n(\Gamma_\omega) = 2|E| \}. \quad (4)$$

#### 4. Effective resistance metric

Let  $\Gamma_\omega$  be a weighted graph with set of vertices  $V$ , and let  $W$  be a non-empty subset of  $V$ . It can be proved that for any function  $x : W \rightarrow \mathbb{R}$  there exists a unique function  $\tilde{x} : V \rightarrow \mathbb{R}$  such that  $\tilde{x}(w) = x(w)$  for all  $w \in W$  and  $(L_{\Gamma_\omega})(v) = 0$  for all  $v \in V \setminus \omega$ . The function  $\tilde{x}$  is the *harmonic extension* of  $x$ . The existence and uniqueness of the harmonic extension of any given  $x$  is a well known fact (e.g. Doyle and Snell, 1984; Kigami, 2001).

**Definition 4.1.** (Effective resistance) Let  $\Gamma_\omega$  be a weighted graph. The *effective resistance* of  $\Gamma_\omega$  is the function  $\rho : V \times V \rightarrow \mathbb{R}$  defined as follows: for any pair of distinct vertices  $u, v \in V$  let  $x : \{u, v\} \rightarrow \mathbb{R}$  be given by  $x(u) = 1$  and  $x(v) = 0$ , and denote by  $\tilde{x}$  its harmonic extension. Define

$$\rho(u, v) = \frac{1}{\langle L_{\Gamma_\omega} \tilde{x}, \tilde{x} \rangle}.$$

Also, set  $\rho(u, u) = 0$  for every  $u \in V$ .

The effective resistance is a metric on  $V$  (e.g. Klein and Randić, 1993; Strichartz, 2006), and for this reason it is known as the *effective resistance distance* or simply *resistance distance* of the graph. It is immediate from the definition that

$$\rho(u, v) = \frac{1}{(L_{\Gamma_\omega} \tilde{x})(u)} = -\frac{1}{(L_{\Gamma_\omega} \tilde{x})(v)}.$$

Other several alternative equivalent formulations of the effective resistance appear in the literature (see e.g. Kigami, 2001; Jorgensen and Pearse, 2008; Ghosh et al., 2008).

One notable aspect of the effective resistance is that it has properties that allow it to be computed from geometric considerations. Those properties can be understood from the idea of viewing the graph as an electrical network, which reflects the fact that this quantity was first proposed, many years ago, in such electric circuit settings (Kirchhoff, 1847). These properties indicate what happens with the resistance between electric nodes when they are connected in series and in parallel. The following theorems give precise graph theoretical statements of them:

**Theorem 4.2. (Nodes connected in series)** Let  $u_1, u_2$  and  $v$  be three vertices in the weighted graph  $\Gamma_\omega$ . Suppose that every path that starts on  $u_1$  and ends on  $u_2$  passes through  $v$ . Then

$$\rho(u_1, u_2) = \rho(u_1, v) + \rho(u_2, v).$$

**Theorem 4.3. (Nodes connected in parallel)** Let  $W$  and  $Z$  be subsets of  $V$ , the set of vertices of a weighted graph  $\Gamma_\omega$ . Suppose that  $W \cup Z = V$  and that their intersection  $W \cap Z$  contains exactly two vertices  $v_1$  and  $v_2$ . Denote by  $\Gamma^W$  and  $\Gamma^Z$  the subgraphs of  $\Gamma_\omega$  induced by  $W$  and  $Z$ . Assume further that the induced subgraphs are connected and that there the set of edges of  $\Gamma_\omega$  joining vertices in  $\Gamma^W$  to vertices in  $\Gamma^Z$  is empty. Then

$$\frac{1}{\rho(v_1, v_2)} = \frac{1}{\rho_W(v_1, v_2)} + \frac{1}{\rho_Z(v_1, v_2)},$$

where  $\rho_W$  and  $\rho_Z$  are the resistance distances in the corresponding subgraphs.

Both theorems 4.2 and 4.3 have been well known for a long time (e.g. Doyle and Snell, 1984; Jorgensen and Pearse, 2008). Detailed elementary proofs of both theorems can be consulted in (Menéndez–Conde, 2023), where definition 4.1 is taken as the starting point. From theorem 4.2 it is easy to see that if  $\Gamma_\omega$  is a tree, then the resistance distance  $\rho(u, v)$  is equal to the sum of the reciprocals  $1/\omega_{i,j}$  of the weights of the edges that lie in the only path joining  $u$  and  $v$ . Theorem 4.3 suggests that if we add paths between two vertices, the resistance distance should decrease (or at least should not increase); indeed, the distance in the whole graph equals half the harmonic mean of the distances restricted to the separate subgraphs. We refer to (Spielman, 2019) for a rigorous proof of that fact: the resistance distance does not increase when edges are added or when the weight of an edge is increased.

The above discussion shows that the following conditions are factors that contribute for two vertices to be close to each other, with respect to the resistance distance:

- There are many paths joining the vertices.
- The paths joining the vertices are short (they have few edges).
- The weights on the edges are large.

In view of this, the quantity defined below becomes a reasonable index to measure the global connectedness of the graph.

**Definition 4.4. (Kirchhoff index).** The *total effective resistance* or *Kirchhoff index* of a weighted graph  $\Gamma_\omega$  with set of vertices  $V$  is

$$\mathbf{K}(\Gamma_\omega) = \sum_{u \neq v \in V} \rho(u, v). \quad (5)$$

Note that the sum in (5) involves all unordered pairs of distinct vertices, regardless if there is an edge joining them or not. Given  $n$  vertices, the Kirchhoff index will be small if there are many connections between them. In general, it is more sensitive than the algebraic connectivity, and in particular it will strictly decrease when edges are added to a graph. Indeed, adding an edge between two vertices increases the resistance distance between them, and since for the rest of vertices the resistance distance cannot decrease we have that the sum of distances will increase. In the appendix we share images (generated in Python) of all the connected graphs with 4 and with 5 vertices, arranged by increasing Kirchhoff index, and with Fiedler values included for reference and comparison.

The following theorem, shows that the Kirchhoff index is a spectral invariant. It also hints on why it is likely to expect it to be more sensitive than the Fiedler value: the Kirchhoff index is dependent on the whole spectrum of the weighted graph, and not only one eigenvalue.

**Theorem 4.5. (Klein–Randić 1993)** The Kirchhoff index of a weighted graph  $\Gamma_\omega$  with  $n$  vertices is given by

$$\mathbf{K}(\Gamma_\omega) = n \sum_{k=2}^n \frac{1}{\lambda_k(\Gamma_\omega)}.$$

Observe that theorem 4.5 is the statement that  $\mathbf{K}(\Gamma_\omega)$  equals the trace of the Moore-Penrose pseudoinverse of the Laplacian  $L_{\Gamma_\omega}$ , that is the operator  $Q$  such that  $QL_{\Gamma_\omega}$  and  $L_{\Gamma_\omega}Q$  are the orthogonal projection over  $\ker(L_{\Gamma_\omega})^\perp$ .

From the spectral representation of the Kirchhoff index, for the unweighted complete graph we have

$$\mathbf{K}(K_n) = n - 1.$$

This allows for a simple computation of the resistance distance  $\rho_n$  between any pair of vertices in  $K_n$ . From the symmetry of the graph, it is clear that the distance is the same between any pair, and so:

$$\rho_n(u, v) = \binom{n}{2}^{-1} \mathbf{K}(K_n) = \frac{2}{n}.$$

The total effective resistance for the lollipop graph  $L_{1,n}$  with  $n$  vertices can then be calculated as follows: Let  $\{v_0, \dots, v_n\}$  be the vertices, chosen so that  $\deg(v_0) = 1$ ,  $\deg(v_1) = n$ . Then

$$\begin{aligned} \mathbf{K}(L_{1,n}) &= \mathbf{K}(K_n) + \sum_{k=1}^n \rho(v_0, v_k) \\ &= (n - 1) + 1 + (n - 2)\left(1 + \frac{2}{n}\right) \\ &= 2n - \frac{4}{n}. \end{aligned}$$

The corresponding calculation for the star graph  $S_n$  can be done directly: the distance from the central vertex to all the other vertices is 1, while the distance between any pair of non central vertices is 2. Thus,

$$\mathbf{K}(S_n) = (n - 1) + 2\binom{n - 1}{2} = (n - 1)^2.$$

We can see that, in contrast to what happens with algebraic connectivity, the Kirchhoff index of the lollipop graph with  $n$  vertices is always smaller than that of the star graph  $S_n$ . Even more so, the quotient  $\mathbf{K}(L_{1,n-1})/\mathbf{K}(S_n)$  tends to zero as  $n \rightarrow \infty$ . Thus, according to the total resistance distance, lollipopgraphs are much better connected than star graphs (figure 2).

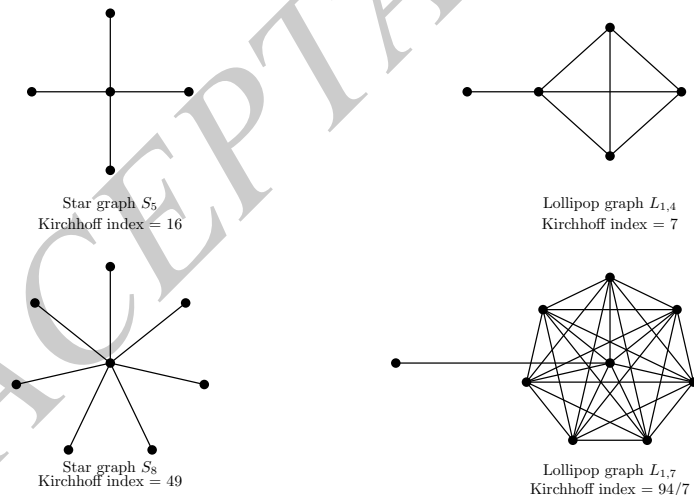


Figure 2: The Kirchhoff index of lollipop graphs is smaller than for the star graphs, in accordance to the existence of more connections, in contrast to what is seen for the Fiedler value.

### 5. Isoperimetric problems

Given any graph  $\Gamma$  with three or more vertices and a constant  $c > 0$ , consider the following questions:

1. Find the maximal value

$$\max\{\lambda_2(\Gamma_\omega) \mid \mathbf{K}(\Gamma_\omega) = c\}.$$

and determine the corresponding weight.

2. Find the minimal value

$$\min\{\mathbf{K}(\Gamma_\omega) \mid \lambda_2(\Gamma_\omega) = c\}.$$

and determine the corresponding weight.

Note that the choice of the constant  $c > 0$  is not essential, since multiplying the weight by a constant  $r > 0$  gives  $\lambda_2(\Gamma_{r\omega}) = r\lambda_2(\Gamma_\omega)$  and  $\mathbf{K}(\Gamma_{r\omega}) = \mathbf{K}(\Gamma_\omega)/r$ .

To determine the absolute algebraic connectivity (aac) of a graph is analogous to the first of the above problems in a number of ways. In both situations, we aim to maximize the Fiedler value under the restriction that some quantity is held fixed. In both cases the fixed quantity is a spectral invariant: for the aac it is the trace of the Laplacian and for our case it is the trace of the pseudoinverse of the Laplacian. Also, the arithmetic mean of the eigenvalues  $\lambda_2, \dots, \lambda_n$  is held fixed for the aac, and their harmonic mean is fixed for the situation considered in this work.

For problem 1, by means of the spectral realization of  $\mathbf{K}(\Gamma_\omega)$  provided by theorem 4.5, we are able to give a very simple proof of the following result:

**Theorem 5.1.** For a positive integer  $n$  and a real constant  $c > 0$ , let  $\mathcal{A}_{n,c}$  be the collection of weighted graphs  $\Gamma_\omega$  with  $n$  vertices and Kirchhoff index  $\mathbf{K}(\Gamma_\omega) = c$ . Then

$$\max\{\lambda_2(\Gamma_\omega) \mid \Gamma_\omega \in \mathcal{A}_{n,c}\} = \frac{n(n - 1)}{c} \tag{6}$$

and this maximal value is attained for the complete graph  $K_n$  with weight equal to the constant  $(n - 1)/c$ .

*Proof.* Suppose that  $\Gamma$  is a graph with  $n$  vertices and that  $\omega'$  is a weight such that  $\lambda_2(\Gamma_{\omega'}) > n(n - 1)/c$ . Then, the other eigenvalues  $\lambda_3, \dots, \lambda_n$  also satisfy  $\lambda_k(\Gamma_{\omega'}) > n(n - 1)/c$ . Then,

$$\mathbf{K}(\Gamma_{\omega'}) = n \left( \frac{1}{\lambda_2(\Gamma_{\omega'})} + \dots + \frac{1}{\lambda_n(\Gamma_{\omega'})} \right) < c.$$

This means that  $\Gamma_{\omega'} \notin \mathcal{A}_{n,c}$ , and hence the maximal value in (6) cannot exceed  $n(n - 1)/2$ . It remains to verify that equality is attained for the graph proposed in the statement of the theorem. So, for  $\Gamma = K_n$  let  $\omega$  be such that every edge weight is equal to  $(n - 1)/c$ . We have

$$\mathbf{K}(\Gamma_\omega) = \frac{c}{n - 1} \mathbf{K}(K_n) = c,$$

so that  $\Gamma_\omega \in \mathcal{A}_{n,c}$ , as needed. On the other hand, we also have that

$$\lambda_2(\Gamma_\omega) = \frac{n - 1}{c} \lambda_2(K_n) = \frac{n(n - 1)}{c}.$$

This concludes the proof of the theorem. □

The simplicity of the proof above may be misleading: in general, it is not straightforward to find the solution to problem 1 even for very simple graphs (see Cruz Ortega and Menéndez-Conde, 2023 for  $P_3$  and  $P_4$ ).

For the second problem, we obtained the following result.

**Theorem 5.2.** *Let  $T$  be a tree with  $n \geq 3$  vertices. For  $c > 0$ , define*

$$X_c = \{\mathbf{K}(T_\omega) \mid \lambda_2(T_\omega) = c\}, \quad \alpha = \inf X_c.$$

Then  $\alpha \notin X_c$ .

*Proof.* Given a tree  $T$  and a fixed constant  $c > 0$ , let  $X_c$  and  $\alpha$  be as in the statement of the theorem. We will show that  $\alpha = n/c$ , from which the stated result clearly follows.

Let  $\{v_0, v_1, \dots, v_{n-1}\}$  be the vertices of  $T$  and without loss of generality suppose that  $v_0$  has degree 1, and that  $v_0 \sim v_1$ . For  $R > 0$  define the weight  $\omega_R$  by

$$\omega_{v_0, v_1} = 1, \quad \omega_{v_i, v_j} = R, \quad i, j \geq 1.$$

Applying the Courant-Fischer inequality to the vector

$$\left(1, -\frac{1}{n-1}, -\frac{1}{n-1}, \dots, -\frac{1}{n-1}\right)$$

we obtain

$$\lambda_2(T_{\omega_R}) \leq \frac{\left(1 + \frac{1}{n-1}\right)^2}{1 + \frac{n-1}{(n-1)^2}} = \frac{n}{n-1}. \tag{7}$$

Take

$$\omega'_R = \frac{c}{\lambda_2(T_{\omega_R})} \omega_R,$$

so that  $\lambda_2(T_{\omega'_R}) = c$  and therefore  $\mathbf{K}(T_{\omega'_R}) \in X_c$  for every choice of  $R$ . Define  $T^0$  as the graph obtained by removing  $v_0$  from  $T$ , and let  $\rho(\cdot, \cdot)$  denote the resistance distance with respect to the weight  $\omega'_R$ . From the definition of the Kirchhoff index, it is clear that

$$\mathbf{K}(T_{\omega'_R}) = \mathbf{K}(T_{\omega'_R}^0) + \sum_{i=1}^{n-1} \rho(v_0, v_i). \tag{8}$$

so that  $\mathbf{K}(T_{\omega'_R}^0) \rightarrow 0$  when  $R \rightarrow \infty$ .

Now, since the weight  $\omega'_R$  is constant on  $T^0$ , we have

$$\mathbf{K}(T_{\omega'_R}^0) = \frac{\lambda_2(T_{\omega_R})}{cR} \mathbf{K}(T^0).$$

For the resistance distances, we have

$$\rho(v_0, v_i) = \frac{\lambda_2(T_{\omega_R})}{c} + c_i \frac{\lambda_2(T_{\omega_R})}{cR},$$

where  $c_i$  denotes the number of edges in the path joining  $v_1$  and  $v_i$ . Substitution in (8) yields

$$\mathbf{K}(T_{\omega'_R}) = \frac{\lambda_2(T_{\omega_R})}{c} \left( \frac{\mathbf{K}(T^0)}{R} + (n-1) + \frac{1}{R} \sum_{i=1}^{n-1} c_i \right).$$

So, using (7) we obtain

$$\mathbf{K}(T_{\omega'_R}) \leq \frac{n}{c} + o(R), \quad R \rightarrow \infty. \tag{9}$$

Also, from theorem 4.5 we have

$$\mathbf{K}(T_{\omega'_R}) > \frac{n}{\lambda_2(T_{\omega'_R})} = \frac{n}{c}.$$

We can conclude

$$\frac{n}{c} < \mathbf{K}(T_{\omega'_R}) < \frac{n}{c} + o(R), \quad R \rightarrow \infty,$$

from which the result follows.  $\square$

An easy consequence of the above result is that for any given tree, weights on the edges can be chosen so that the quotient  $\lambda_3/\lambda_2$  is arbitrarily large:

**Corollary 5.3.** *Let  $T$  be a tree with three or more vertices and  $r > 0$ . A weight  $\omega$  can be assigned so that  $\lambda_3/\lambda_2 > r$ .*

*Proof.* With the notation of theorem 5.2, from the proof of the theorem we know that

$$\frac{n}{c} = \inf X_c.$$

This means that for every  $\epsilon > 0$ , there exist a weight  $\omega$  such that  $\lambda_2(T_\omega) = c$  and

$$\mathbf{K}(T_\omega) - \frac{n}{c} < \epsilon.$$

But then, we have

$$0 < \frac{n}{\lambda_3(T_\omega)} < \frac{n}{\lambda_2(T_\omega)} + \dots + \frac{n}{\lambda_n(T_\omega)} < \epsilon.$$

So, for  $\lambda_2 = c$  fixed we can choose weights for which  $1/\lambda_3$  is arbitrarily small, as we wanted.  $\square$

## References

Abreu, N. M. (2007). Old and new results on algebraic connectivity of graphs. *Linear Algebra and its Applications*, 423(1):53–73.

Alon, N. and Milman, V. D. (1985).  $\lambda_1$ , isoperimetric inequalities for graphs, and superconcentrators. *Journal of Combinatorial Theory, Series B*, 38(1):73–88.

Biyyıkođlu, T. and Leydold, J. (2009). Algebraic connectivity and degree sequences of trees. *Linear Algebra and its Applications*, 430(2–3):811–817.

Chung, F. R. K. (2007). Four Cheeger-type inequalities for graph partitioning algorithms. In *Proceedings of the International Congress of Chinese Mathematicians (ICCM), Vol. II*, pages 751–772.

Courant, R. and Hilbert, D. (1937). *Methods of Mathematical Physics, Vol. I*. Interscience Publishers (John Wiley & Sons), New York.

Cruz Ortega, A. and Menéndez-Conde, F. (2023). Conectividad algebraica máxima para caminos con resistencia total fija. *Pesquimat*, 26(2):11–24.

Cvetkovic, D., Rowlinson, P., and Simić, S. (2009). *Introduction to the Theory of Graph Spectra*, volume 75 of *London Mathematical Society Student Texts*. Cambridge University Press.

de Lima, L. S., de Abreu, N. M. M., Oliveira, C. S., and de Freitas, M. A. A. (2007). Laplacian integral graphs in  $s(a,b)$ . *Linear Algebra and its Applications*, 423:136–145.

Doyle, P. G. and Snell, D. J. (1984). *Random Walks and Electrical Networks*. Carus Mathematical Monographs. MAA.

Ellens, W., Spieksma, F. M., van Mieghem, P. F. A., Jamakovic, A., and Kooij, R. E. (2011). Effective graph resistance. *Linear Algebra and its Applications*, 435(10):2491–2506.

Fallat, S. and Kirkland, S. (1998). Extremizing algebraic connectivity subject to graph theoretic constraints. *Electronic Journal of Linear Algebra*, 3:48–74. PDF (mirror): <https://emis.de/ft/34224>.

Fiedler, M. (1973). Algebraic connectivity of graphs. *Czechoslovak Mathematical Journal*, 23:298–305.

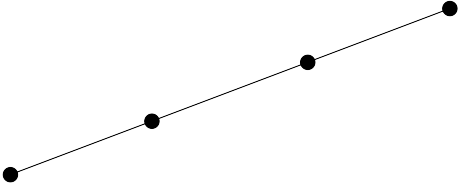
Fiedler, M. (1989). Laplacian of graphs and algebraic connectivity. *Banach Center Publications*, 25(1):57–70.

Fiedler, M. (1990). Absolute algebraic connectivity of trees. *Linear and Multilinear Algebra*, 26(1-2):85–106.

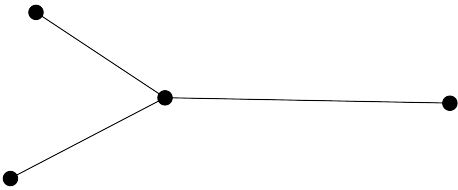
- Ghosh, A., Boyd, S., and Saberi, A. (2008). Minimizing effective resistance of a graph. *SIAM Review*, 50(1):37–66.
- Jorgensen, P. E. T. and Pearse, E. P. J. (2008). Operator theory of electrical resistance networks. arXiv preprint arXiv:0806.3881 [math.OA].
- Kigami, J. (1994). Effective resistances for harmonic structures on p.c.f. self-similar sets. *Mathematical Proceedings of the Cambridge Philosophical Society*, 115(2):291–303.
- Kigami, J. (2001). *Analysis on Fractals*. Cambridge Tracts in Mathematics. Cambridge University Press.
- Kirchhoff, G. (1847). Über die Auflösung der Gleichungen, auf welche man bei der Untersuchung der linearen Vertheilung galvanischer Ströme geführt wird. *Annalen der Physik und Chemie*, 72:497–508.
- Klein, D. and Randić, M. (1993). Resistance distance. *Journal of Mathematical Chemistry*, 12(1):81–95.
- Menéndez-Conde, F. (2023). La métrica de resistencia efectiva. *Miscelánea Matemática*, 76:45–66.
- Mohar, B. (1992). Laplace eigenvalues of graphs—a survey. *Discrete Mathematics*, 109(1):171–183.
- Powers, R. T. (1976). Resistance inequalities for the isotropic Heisenberg ferromagnet. *Journal of Mathematical Physics*, 17(10):1910–1918. MR 0436847.
- Predari, M., Berner, L., Kooij, R., and Meyerhenke, H. (2023). Greedy optimization of resistance-based graph robustness with global and local edge insertions. *Social Network Analysis and Mining*, 13:130.
- Rojo, O., Medina, L., de Abreu, N. M. M., and Justel, C. (2010). Extremal algebraic connectivities of certain caterpillar classes and symmetric caterpillars. *Electronic Journal of Linear Algebra*, 20:136–157.
- Spielman, D. A. (2019). Spectral and algebraic graph theory. Lecture notes, Yale University.
- Strichartz, R. S. (2006). *Differential Equations on Fractals: A Tutorial*. Princeton University Press.
- Yang, C., Mao, J., Qian, X., and Wei, P. (2019). Designing robust air transportation networks via minimizing total effective resistance. *IEEE Transactions on Intelligent Transportation Systems*, 20(6):2353–2366.
- Zhou, X., Zehmakan, A. N., and Zhang, Z. (2025). Efficient algorithms for minimizing the Kirchhoff index via adding edges. *IEEE Transactions on Knowledge and Data Engineering*. Preprint available as arXiv:2503.21409.

Appendix A.

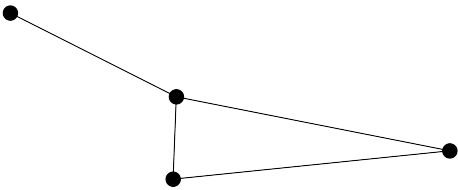
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F value = 0.586



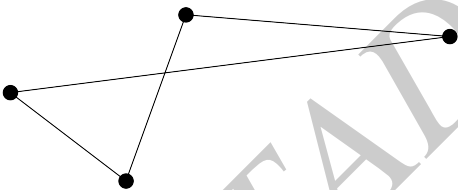
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F value = 1.000



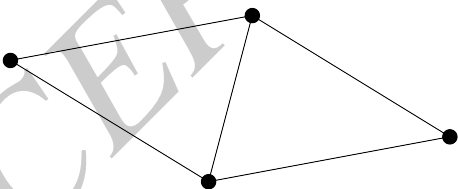
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F value = 1.000



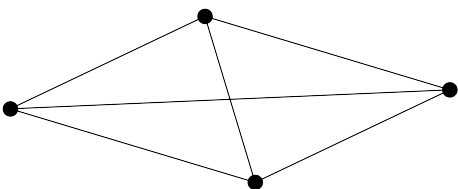
#4 K index = 5.00  
F value = 2.000



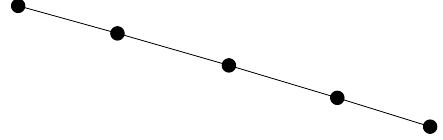
#5 K index = 4.00  
F value = 2.000



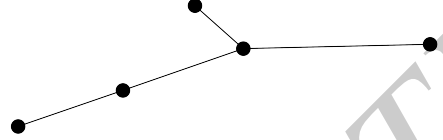
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F value = 4.000



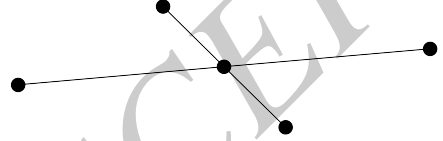
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F value = 0.382



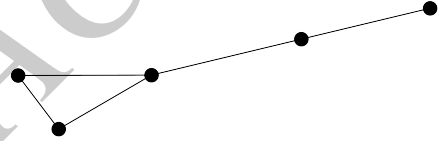
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F value = 0.519



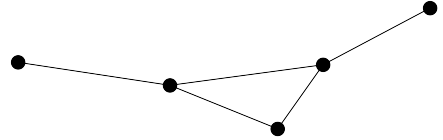
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F value = 1.000



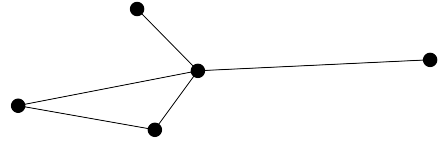
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F value = 0.519



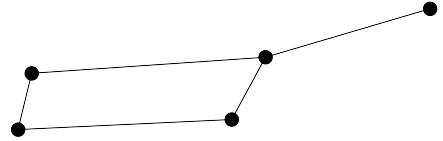
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F value = 0.697



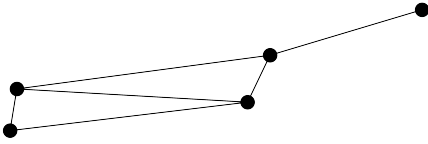
#6 K index = 12.67  
F value = 1.000



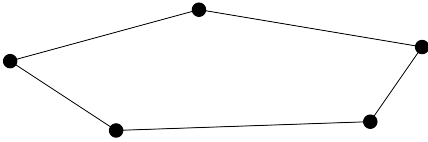
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F value = 0.830



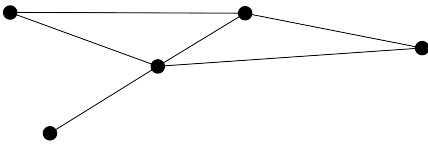
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F value = 0.830



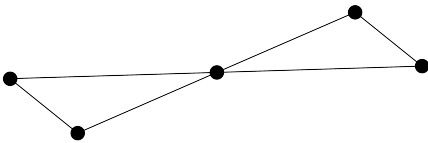
#13 K index = 10.00  
F value = 1.382



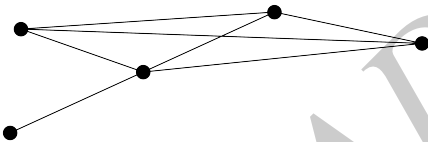
#14 K index = 9.75  
F value = 1.000



#15 K index = 9.33  
F value = 1.000



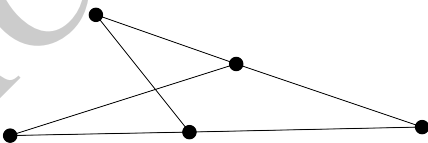
#16 K index = 8.50  
F value = 1.000



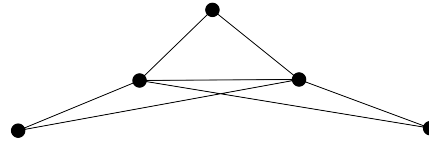
#17 K index = 8.18  
F value = 1.382



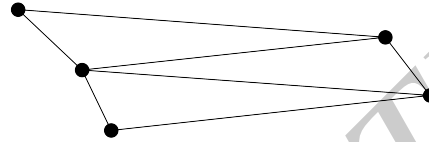
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F value = 2.000



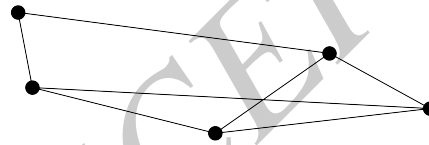
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F value = 2.000



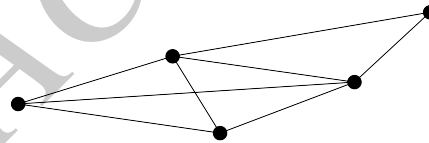
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F value = 1.586



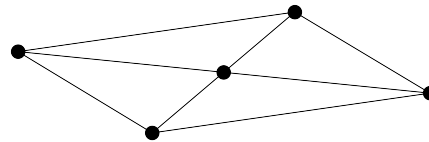
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F value = 2.000



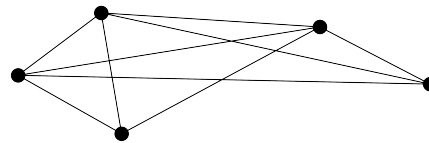
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F value = 2.000



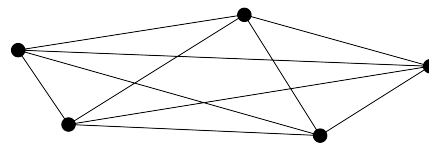
#27 K index = 5.33  
F value = 3.000



#28 K index = 4.67  
F value = 3.000



#29 K index = 4.00  
F value = 5.000



The six connected (unweighted) graphs with four vertices and the twentyone connected graphs with five vertices. The graphs are arranged by increasing Kirchhoff index. Note that in the 4 vertices case, they happen to also be arranged in non

increasing order with respect to the Fiedler value; however, that of the Fiedler value is reversed in some cases. is not the case for graphs with 5 vertices, for which but the size

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