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# A quick overview on initial enlargement of filtrations

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**Resumen**

En este trabajo presentamos, a través de un ejemplo particular, una breve introducción al tema de engrosamiento de filtraciones, en particular, el caso inicial, es decir, cuando expandimos una filtración con una $\sigma-$algebra.

**Abstract**

We describe, thought of a particular example, the meaning of initial enlargement of filtration, this means, when we expand a filtration with a $\sigma-$algebra.

*Palabras Clave:* Filtration, Stochastic processes, Brownian bridge

## 1. Introduction

Whenever we want to model a random experiment, we need to define a probability space $(\Omega, \mathcal{F}, P)$ where the element $\mathcal{F}$ is a $\sigma-$algebra. This structure contains the events for which it is possible to calculate its probability. Now, consider $\mathcal{F}_1$ and $\mathcal{F}_2$, both sub $\sigma-$algebras, and suppose that $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}$. This means that $\mathcal{F}_2$ contains more events than $\mathcal{F}_1$. For example, if $X_n$ represents the result of the $n-$th flip of a coin and $A$ is the event to get two heads, then $A \in \mathcal{F}_2$, however, $A \notin \mathcal{F}_1$.

This motivates us to consider a filtration. A filtration is an infinite increasing succession of $\sigma-$algebras contained in $\mathcal{F}$, which have the historical information, but no the future information, of a random phenomenon.

By enlargement of filtration, we refer that we expand the filtration with additional information until we get another filtration. The study of enlargement of filtration began with the work of Itô in 1976 (see [16]), when he gave meaning to the integral

$$\int_0^t W_1 dW_s,$$

expanding the natural filtration of $W$ with the variable $W_1$ and proving that the integral in this new filtration was well defined.

Two years later, in 1978, Barlow (see [4]) studied the problem of how to enlarge the filtration, in a minimal way, when $L$ is a positive random variable. More precisely, how to make $L$ a stopping time (see Definition 2.9) and preserve the well-definedness of the stochastic integrals. Since then, many studies about this topic have been addressed, see, for instance, [19], [20] or the related chapters of the books [25] and [26].

The results of enlargement of filtrations have been extensively used in finance to study the consequences of insider trading (see the references of Section 4).

In this work, we will make a brief introduction to this topic through a simple example. Firstly, we review some basic definitions of stochastic processes in Section 2. Then, in Section 3, we will enlarge the filtration with the $\sigma-$algebra generated by a brownian motion evaluated $t = 1$ and, from this, we will explain the basic ideas of expanding a filtration. Finally, in Section 4, we present a simple example of application.

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## 2. Basic concepts

In this section, we will make a review of basic concepts which will be needed in the remainder of this paper.

### 2.1. Stochastic processes and filtrations

A probability space $(\Omega, \mathcal{F}, P)$ is a shortlist of three elements. Here, $\Omega$ is a nonempty set, $\mathcal{F}$ is a $\sigma-$algebra of subsets of $\Omega$ and $P$ is a probability mesure. The elements of $\Omega$ represent all the possible results in a random experiment, while those in $\mathcal{F}$ are the possible events of the experiment.

**Definition 2.1.** *A random variable is a function $X : \Omega \to \mathbb{R}$ such that $X^{-1}(\mathcal{B}(\mathbb{R})) \subset \mathcal{F}$, where $\mathcal{B}(\mathbb{R})$ is the $\sigma-$algebra of Borel.*

As an example, let us consider the experiment of throwing a coin. Let X be the random variable that takes the value 1 if the result is head, and 0 if the result is tail. For this experiment, we have $\Omega = \{\text{head}, \text{tail}\}$ and $\mathcal{F} = \{\emptyset, \Omega, \{\text{head}\}, \{\text{tail}\}\}$.

**Definition 2.2.** *Let $(E, \mathcal{B})$ be a measurable space. A stochastic process, $X = \{X_t\}_{t\geq 0}$, is a family of random variables, such that $X_t : (\Omega, \mathcal{F}) \to (E, \mathcal{B})$, i.e., it is considered as function of two variables so that for each couple $(t, \omega)$ corresponds $X(t, \omega)$. For each $\omega \in \Omega$ fixed, the function $t \to X_t(\omega)$ is called the path of the process.*

These processes represent random phenomena that evolve in time, for example, the capital of an insurance company, the price of a stock, or the number of cars passing through a toll booth in any given period of time. When we observe a random phenomenon, we can be interested in the events that occur before or inclusive until instant $t$. For all fixed $t$, these events form a $\sigma-$algebra that is contained in $\mathcal{F}$. Thus, if we are interested in studying random phenomena and we want to include the time factor, we can add to our probability space a filtration.

**Definition 2.3.** *A filtration of $\mathcal{F}$ is a non decreasing family of $\sigma-$algebras $\{\mathcal{F}_t\}_{t\geq 0}$, namely,*

$$\mathcal{F}_s \subset \mathcal{F}_t, \quad \forall s, t > 0 \text{ such as } s < t.$$

Note that the non-decreasing condition makes sense due to the fact that, as time evolves, we have more information about the random phenomenon.

**Definition 2.4.** *A stochastic processes X is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if $X_t$ is $\mathcal{F}_t-$measurable $\forall\, t \geq 0$.*

This means that the process can not anticipate the future or, in other words, the possible events up to a given instant $t$ only depend on the past events. In particular, note that whether a process is measurable or not, only depends on $\mathcal{F}$, while to be adapted depends on the filtration. In general, measurable does not imply adapted.

**Definition 2.5.** *The filtration $\mathcal{F}^X$ is defined by*

$$\mathcal{F}_t^X = \sigma(X_s; s \leq t),$$

*and it is called filtration generated by X.*

If $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration in a probability space $(\Omega, \mathcal{F}, P)$, we can associate to it other possible filtrations, for example:

**1** $\mathcal{F}_{t^+} = \cap_{s>t}\mathcal{F}_s$, $t \geq 0$.

**2** If the probability space is complete, we can denote $\overline{\mathcal{F}_t}$ the smallest $\sigma-$algebra that contains to $\mathcal{F}_t$ and all elements of $\mathcal{F}$ of probability equals to zero.

If $\mathcal{F}_t = \mathcal{F}_{t^+}$ for each $t$, it is said the filtration is *right continuous*. We say that a filtration satisfies the usual conditions if it is right continuous and $\mathcal{F}_t = \overline{\mathcal{F}_t}$.

2.2. *Some examples of stochastic processes*

The different types of stochastic processes are obtained by considering the different characteristics, mainly, the relations of dependence between the random variables that make up the process. For the purposes of this work, we will focus on defining martingales, semimartingales and we will discuss about two important processes: the brownian motion and the brownian bridge.

We begin with martingales. These processes have a simple interpretation in terms of fair games: if $X_s$ is the capital of a gamer at time $s$, the expected value (see Definition 2.6) tells us that the average fortune at the future time $t$, given that the history of the game is known until time $s$, is its capital at time $t$, that is, the game is fair because on average the player does not lose or win.

**Definition 2.6.** *An adapted process X is called a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, if $E(|X_t|) < \infty$, and if $s \leq t$ then $E(X_t|\mathcal{F}_s) = X_s$.*

If a process $X$ is a martingale, this means that its expected value, in a future time $t$, is the value of the process in the last moment when it was observed.

**Definition 2.7.** *When the sample paths of a stochastic process X are, almost sure, right continuous with left limits it is said that X is càdlàg.*

**Definition 2.8.** *If X is a càdlàg martingale and, in addition, $E(|X_t|^2) < \infty$ for all $t \geq 0$, we will say that X is an square integrable martingale.*

To define the next concept, we need some additional notation. If $T$ is a nonnegative random variable and $Y_t$ is any process, we define

$$Y_t^T = \begin{cases} Y_{T\wedge t} & \text{in } \{T > 0\}, \\ 0 & \text{in } \{T = 0\}, \end{cases} \tag{1}$$

where $T \wedge t$ stands for the minimum between $T$ and $t$.

**Definición 2.9.** *A stopping time $\tau$ is a random variable such that, for each $t \geq 0$, it is verified that $\{\tau \leq t\} \in \mathcal{F}_t$.*

A stopping time registers the moment in which an event of the process occurs. More precisely, at any time $t$, we can know if this event has occurred or not.

**Definition 2.10.** *A process X is a local martingale with respect to $\{\mathcal{F}_t\}_{t\geq 0}$, if there are stopping times $\{T_n\}$ such that $T_n \uparrow \infty$ so that $X^{T_n}$ is a martingale with respect to $\{\mathcal{F}_{t\wedge T_n}\}_{t\geq 0}$.*

Local martingales are important for many reasons, one of those being for the integrability of stochastic proceses.

The classic Itô integral [17] is defined for *adapted* processes to the filtration for which the integrator is a brownian motion (see Definition 2.13). The idea of definition of this integral is similar to the integral of a function with respet to a measure. First, it is defined for simple processes and then, by approximation, for more general processes. The interested reader can consult [2] and [21] for a detailed exhibition about the construction of this integral.

**Remark 2.11.** *The idea of Itô integral has been generalized to more general integrators as local martingales and processes of finite variation, namely those processes whose sample paths are functions of finite variation (see Definition 2.17). In addition, it can be shown (it is beyond the scope of these notes, though) that linear combinations of those are, in a sense, all processes that can be used as integrators. These processes are called "good integrators" or semimartingales, reducing the integrability requirements.*

**Definition 2.12.** *A processes $X$ is said to be a continuous semimartingale if $X$ can be written as $X = M + A$ ,where $M$ is a continuous local martingale and $A$ is a càdlàg adapted process with finite variation on* $[0, t]$ *for* $t > 0$.

For a detailed exposition of the previous concepts, as well as of those of stochastic integration, the interested reader can consult [7], [12], [25] and the references therein.

### 2.3. Brownian motion

Now, we give a brief introduction to the brownian motion $W$ (see for instance [21], [12], [26], [27]), one of the most important processes in stochastic calculus due to its applications in different disciplines of science (we refer the interested reader to [3], [23]). It was observed, for the first time in 1828, by the botanist Robert Brown (see [8],[9]), when he studied the movement of pollen grains suspended in a certain substance. Based on Brown's observations, several theories emerged, but it was Einstein [13], in 1905, who gave the correct explanation. The first rigorous mathematical construction of brownian motion is due to Wiener [28].

**Definition 2.13.** *A brownian motion $W = \{W_t\}_{t\geq 0}$ is an adapted stochastic process with continuous paths that satisfies the following properties:*

**1** $W_0 = 0$ *with probability* 1.

**2** *For $s \leq t$, $W_t - W_s$ is independent of $\mathcal{F}_s$.*

**3** *For $s \leq t$, $W_t - W_s$ has a normal distribution with mean equal to* 0 *and variance $t - s$.*

In what follows, we will mention only the properties of Brownian motion that we will be useful in this work.

**Proposition 2.14.** *The brownian motion is an square integrable martingale.*

*Demostración.* From Property 2 of Definition 2.13, we see

$$E(W_t - W_s|\mathcal{F}_s) = E(W_t - W_s) = 0.$$

□

**Proposition 2.15.** *The quadratic variation of brownian motion in the interval* $[0, t]$ *is* $t$.

In order to establish the proof of this result, it is necessary to give some definitions.

**Definition 2.16.** *The collections of points $P = \{t_0, \dots t_n\}$ is a partition of an interval $[a, b]$ if $a = t_0 < t_1 < \cdots < t_n = b$ holds.*

**Definition 2.17.** *Let $f : [a, b] \to \mathbb{R}$ be a function. The variation $V_f$ of $f$ over $[a, b]$ is defined to be*

$$V_f = \sup\left\{\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| : \right. \\ \left. P = \{t_0, \dots t_n\} \text{ is a partition of } [a, b]\right\}.$$

Now, we proceed with the proof of Proposition 2.15.

*Demostración.* Let $\Delta_n$ be a succession of partitions of $[0, t]$, such that

$$||\Delta_n|| = \text{máx}\{t_i - t_{i-1}, i = 1, \dots n\} \to 0.$$

Then,

$$\begin{aligned} E(\Sigma_i(W_{t_i} - W_{t_{i-1}})^2 - t)^2 &= E(\Sigma_i(W_{t_1} - W_{t_{i-1}})^2 - \Sigma_i(t_t - t_{i-1}))^2 \\ &= E\Big(\Sigma_i[(W_{t_i} - W_{t_{i-1}})^2 - (t_i - t_{i-1})]^2 \\ &\quad + 2\,\Sigma_{j,k}[(W_j - W_{t_{j-1}})^2 - (t_j - t_{j-1})]\times \\ &\quad [(W_k - W_{t_{k-1}})^2 - (t_k - t_{k-1})]\Big) \\ &= \Sigma_i E[(W_{t_i} - W_{t_{i-1}})^2 - (t_i - t_{i-1})]^2 \\ &= \Sigma_i E(W_{t_i} - W_{t_{i-1}})^4 + \Sigma_i(t_i - t_{i-1})^2 \\ &= 4\Sigma_i(t_i - t_{i-1})^2 \\ &\leq 4||\Delta_n||\Sigma(t_i - t_{i-1}) = 4||\Delta_n||t \to 0 \quad if \quad n \to \infty. \end{aligned}$$

□

The following theorem is known as Levy theorem and establishes the conditions that characterize brownian motion. Its proof can be found, for instance in, [7], [12].

**Theorem 2.18.** *Let $X$ be a continuous local martingale such that $\{X_t^2 - t\}_{t\geq 0}$ is a local martingale and $X_0 = 0$. Then, $X$ is a brownian motion with respect to the filtration for which $X$ is a local martingale.*

**Remark 2.19.** *Formally speaking, the quadratic variation of a process $X$ is a stochastic process which is denoted as $[X]$ and it is defined by*

$$[X]_t = \lim_{n\to\infty}(L^1)\Sigma_i^\infty(X_{t_{i-1}^n \wedge t} - X_{t_i^n \wedge t})^2, \quad t \geq 0.$$

*As a consequence, Proposition 2.15 establishes that $[W]_t = t$.*

### 2.4. Brownian Bridge

This process is derived from brownian motion by requiring an extra constraint and it is used, for example, in Statistics to derivate the Kolmogorov-Smirnov test (the interested reader can consult [11], [22]).

**Definition 2.20.** *The brownian bridge $\{X_t\}_{0\leq t\leq 1}$ is a Gaussian process (i.e., for any finite sub family $(t_0, t_1, \dots, t_n) \in \mathbb{R}^n$, the random vector $(X_{t_0}, \dots X_{t_n})$ has a multivariate normal distribution) that satisfies the following:*

**1** *$X_0 = 0$ and $X_1 = 0$ with probability* 1.

**2** $E(X_t) = 0$

**3** *For $s \leq t$, $cov(X_s, X_t) = s(1-t)$.*

A natural question is whether such a process exists. The answer is yes. We observe the following result.

**Proposition 2.21.** *Let $W$ be a brownian motion. The process $X_t = W_t - tW_1$ is a brownian bridge.*

*Demostración.*

$$E(X_t) = E(W_t - tW_1) = E(W_t) - tE(W_1) = 0.$$

if $s \leq t \leq 1$.

$$\begin{aligned} \text{cov}(X_s, X_t) &= E(X_s X_t) = E((W_s - sW_1)(W_t - tW_1)) \\ &= E(W_s W_t) - sE(W_1 W_t) - tE(W_s W_1) + stE(W_1 W_1) \\ &= s - st - ts + st = s(1-t). \end{aligned}$$

□

Proposition 2.21 give us a continuous version of the brownian bridge. In [26], the reader can see how the brownian bridge may be viewed as brownian motion conditioned to $W_1 = 0$, namely $\{W_t, 0 \leq t \leq 1 | W_1 = 0\}$.

We may consider the brownian bridge between 0 and $y$ for $0 \leq t \leq 1$ by setting,

$$X_t = W_t - t(W_1 - y).$$

And more generally, between $x$ and $y$ for $0 \leq t \leq T$, by expressing,

$$X_t = x + W_t - \frac{t}{T} W_T + \frac{t}{T}(y - x).$$

For Section 4, we will need the following concept.

### 2.5. *Stochastic differential equations*

We consider the non-autonomous stochastic differential equation written formally as

$$\begin{aligned} dX_t &= b(t, X_t)dt + \sigma(t, X_t)dW_t, \quad 0 < t \leq T, \\ X_0 &= x_0. \end{aligned}$$

The interpretation of this equation is

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad 0 \leq t \leq T. \quad (2)$$

Here, $W$ is a real-valued Brownian motion defined on the completed probability space $(\Omega, \mathcal{F}, P)$, equipped with the filtration $(\mathcal{F}_t : 0 \leq t \leq T)$ that satisfies the usual conditions. The coefficients $b, \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ are measurable functions and are named drift and diffusion coefficients, respectively, and $x_0 \in \mathbb{R}$ stands for the initial condition. In equation (2), the stochastic integral is defined in Itô's sense. The process $X = \{X_t : t \geq 0\}$ that solves (2) is called Itô's process and its coefficients satisfy certain regularity conditions, i.e., $\sigma$ belongs to the class of processes such that: i) for $t \in (0, T]$ the relation $(s, \omega) \to \sigma_s(\omega)$ defined in $(0, t] \times \Omega$ is measurable respect of the $\sigma-$algebra $\mathcal{B}_{[0,t]} \times \mathcal{F}_t$, where $\mathcal{B}_{[0,t]}$ is the Borel $\sigma$-algebra on $[0, t]$, and ii) $P\left(\int_0^T \sigma_t^2 dt < \infty\right) = 1$ are satisfied, while $b$ belongs to the class of processes such that i) and ii') $P\left(\int_0^T |b_t| dt < \infty\right) = 1$ hold.

To know about the existence and uniqueness of the solutions of these equations, as well as more details, the reader can consult, for example [2], [12], [24].

## 3. Enlargement of filtrations

As we briefly mentioned in the above section, a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ contains the information related to a given random phenomenon up to the moment $t$. If we suppose the arrival of new information, this induces to consider a different filtration $\{\mathcal{G}_t\}_{t\geq 0}$ such that $\mathcal{F}_t \subset \mathcal{G}_t$ for each $t \geq 0$. For example, in financial markets there are different types of traders whose behavior is induced by different types of information. It is commonly assumed that all traders have the same information. If we assume that the market can accept differences in information between traders, then we will have to consider a new filtration to approach this problem (see, for instance [10] and references therein).

The enlargement of filtration is split into two cases: initial enlargements, for which, $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(L)$ and progressive enlargement, where $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$. In both cases, $\sigma(L)$ and $\mathcal{H}_t$ are assumed to represent the new information.

When a filtration, is enlarged some questions need to be answered: in which cases a semimartingale remains a semimartingale and what is the new descomposition in $\mathcal{G}$.

We want to focus our interest in inicial enlargement. For such reason, rather than studying the general problem, we discuss an interesting problem.

### 3.1. *Brownian Bridge*

Let us begin with a brownian motion $W = \{W_t\}_{t\geq 0}$ defined in a complete space of probability $(\Omega, \mathcal{F}, P)$ that satisfies the usual conditions and it is equipped with the natural filtration $\mathcal{F}_t^W$. Now, we define a new filtration that satisfies the usual conditions as $\mathcal{G}_t = \bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon}^W \vee \sigma(W_1)$.

We know that $W$ is a martingale in $\mathcal{F}_t$, (see Proposition 2.14) however it is not a martingale in the filtration $\mathcal{G}_t$. In fact, let be $t \leq 1$ and note that $W_1$ is measurable with respect to $\mathcal{G}$, then

$$E(W_1 | \mathcal{G}_t) = E(W_1 | \mathcal{F}_t^W \vee \sigma(W_1)) = W_1 \neq W_t.$$

Nevertheless, $W$ is a semimartingale in $\mathcal{G}_t$. The following result presents us the decomposition of $W$ as $\mathcal{G}-$ semimartingale.

**Proposition 3.1.** *The process $B$ which is defined as*

$$B_t = W_t - \int_0^{t\wedge 1} \frac{W_1 - W_u}{1 - u} du, \quad (3)$$

*is a $\mathcal{G}_t-$martingale and a brownian motion in this same filtration.*

*Demostración.* We begin by computing the following expectation. Let $0 \le s < t < 1$. By means of the property of independent increments, we have

$$\begin{aligned} E(W_t - W_s|W_1 - W_s) =& E(W_{t-s}|W_{1-s}) \\ =& E\left(W_{t-s} - (t-s)\frac{W_{1-s}}{1-s}|W_{1-s}\right) \\ &+ \frac{t-s}{1-s}E(W_{1-s}|W_{1-s}). \end{aligned}$$

Notice that the brownian bridge $X_{t-s}$ is independent of $W_{1-s}$, indeed, $\mathrm{cov}(W_{t-s} - \frac{t-s}{1-s}W_{1-s}, W_{1-s}) = 0$ and we know that gaussian variables are independent if they have zero covariance. Thus,

$$E(W_t - W_s|W_1 - W_s) = \frac{t-s}{1-s}(W_1 - W_s).$$

On the other hand, since $\mathcal{F}_s^W$ is independent of $\{W_{s+h} - W_s\}_{h\ge 0}$ we have,

$$\begin{aligned} E(W_t - W_s|\mathcal{G}_s) =& E(W_t - W_s|W_1 - W_s) \\ =& \frac{t-s}{1-s}(W_1 - W_s). \end{aligned}$$

Now, we compute

$$\begin{aligned} E(B_t - B_s|\mathcal{G}_s) =& E\left(W_t - W_s - \int_s^t \frac{W_1 - W_u}{1-u}du|\mathcal{G}_s\right) \\ =& E(W_t - W_s|\mathcal{G}_s) - E\left(\int_s^t \frac{W_1 - W_u}{1-u}du|\mathcal{G}_s\right) \end{aligned}$$

and by Fubini's Theorem for conditional expectation

$$\begin{aligned} =& E(W_t - W_s|\mathcal{G}_s) - \int_s^t \frac{1}{1-u}E(W_1 - W_u|\mathcal{G}_s)du \\ =& E(W_t - W_s|\mathcal{G}_s) - \int_s^t \frac{1}{1-u}E(W_1 - W_s - (W_u - W_s)|\mathcal{G}_s)du \\ =& E(W_t - W_s|\mathcal{G}_s) - \frac{t-s}{1-s}(W_1 - W_s) \\ =& 0. \end{aligned}$$

Note that we do not have problems in $t = 1$ since

$$E^2(|W_1 - W_s|) \le E((W_1 - W_s)^2) \le c(1-s),$$

for some constant $c$ and for all $s \in [0,1]$, therefore

$$E\left(\int_0^1 \frac{|W_1 - W_s|}{1-s}ds\right) \le \int_0^1 \frac{\sqrt{1-s}}{1-s} < \infty.$$

Accordingly, $E(B_t|\mathcal{G}_s) = B_s$ and $B$ is a $\mathcal{G}$−martingale. Also by properties of the quadratic variation we have $[B]_t = t$ and the Theorem 2.18 gives $B$ is a brownian motion. □

For more details and general version of the results presented in this section the reader can consult [25] and [18].

## 4. Example

The results of enlargement of filtrations have been used in finance to study the consequences of insider trading ([1], [5], [15], [14]). The insider trading is when some participants of the market have material information which they do not share with the rest of the market. The insider knowledge can produce the existence of arbitrage (that is, the possibility to get profits without risk), or changes in the prices dynamic.

For example, in the Black-Scholes model [6], which it is used to determine the value of certain financial assets called European options, the price $S$ of underlaying asset, due to different factors, evolves randomly, i.e., it is represented by the following stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $\mu$ and $\sigma$ are constants. It is assumed that the asset has a constant interest rate $r$. The agent invests his money in an investment portfolio, thus his wealth is

$$dX_t = rX_t dt + \hat{\pi}_t(dS_t - rS_t dt), \ X_0 = 0, \ t \in [0,T]. \quad (4)$$

We assume that $r$ is a constant interest. The first addition of the right side of the equality (4) represents the investment without risk, while in the second adding $\hat{\pi}$ is the number of shares of the risky asset. If we include $\pi = \hat{\pi}S_t/X_t$ as the proportion of wealth invested in the risky asset and $\theta = \frac{\mu - r}{\sigma}$. The above equation is

$$dX_t = (r + \pi_t\sigma\theta)X_t dt + \pi_t\sigma X_t dW_t, \ X_0 = 0, \ t \in [0,T].$$

Using Itô's formula (see [2], [24]) and properties of expectation value it is estimated that in $\pi = \frac{\theta}{\sigma}$ the agent can maximize his expected wealth $E(\ln(X_T))$ and

$$\sup E(\ln(X_T)) = \ln(x) + T\left(r + \frac{\theta^2}{2}\right).$$

Now, we suppose that we enlarge the filtration with $S_1$ or equivalently with $W_1$. Using Proposition 3.1, the dynamics of $S$ and $X$, for $t < 1$, are

$$\begin{aligned} dS_t =& \left(\mu S_t + \frac{W_1 - W_t}{1-t}\sigma S_t\right)dt + \sigma S_t dB_t, \\ dX_t =& X_t(\pi_t\sigma\hat{\theta}_t + r)dt + \sigma\pi_t X_t dB_t. \end{aligned}$$

Here $\hat{\theta} = \frac{\mu - r}{\sigma} + \frac{W_1 - W_t}{1-t}$. Following a similar procedure it is found that

$$\begin{aligned} \max_{\pi \text{ ad. in } \mathcal{G}_t} E(\ln(X_T)) =& \max_{\pi \text{ ad. in } \mathcal{F}_t^W} E(\ln(X_T)) \\ &+ E\left(\int_0^T \left(\frac{W_1 - W_s}{1-s}\right)^2 ds\right) \\ =& \max_{\pi \text{ ad. in } \mathcal{F}_t^W} E(\ln(X_T)) - \frac{1}{2}\ln(1-T), \end{aligned}$$

where with $\pi$ ad. in $\mathcal{H}$, we refer to admissible strategies in the $\sigma$−algebra $\mathcal{H}$, this means that, once the new price of underlaying asset is announced, the investors readjust their portfolio without adding or consuming money. Note that if $T = 1$, the the maximum of the expected wealth is infinity, thus there is an arbitrage opportunity.

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