DESDE 2013
https://repository.uaeh.edu.mx/revistas/index.php/icbi/issue/archive
Pädi Boletín Científico de Ciencias Básicas e Ingenierías del ICBI
ISSN: 2007-6363

Publicación Semestral Pädi Vol. 9 No. 18 (2022) 85–91

On the nonlinear output regulation for systems described by Takagi-Sugeno fuzzy descriptor models with a steady-state mapping as an LMI optimization problem
La regulación no lineal de la salida para sistemas descritos por modelos descriptores tipo Takagi-Sugeno con variedad estacionaria como un problema LMI de optimización

L. Poblete[a], T. Hernández-Cortés[a,*], V. Estrada-Manzo[a]

[a]Department of Mechatronics, Universidad Politécnica de Pachuca, 43830, Zempoala, México.

Resumen

This paper is devoted to provide a numerical solution to the nonlinear output regulation problem for descriptor systems. The control law under design is a nonlinear one, it consists on a nonlinear stabilizer combined with linear steady-state mapping as well as nonlinear steady-state input mapping; all of them are computed via linear matrix inequalities. A numerical example and a mechanical system as well are used to illustrate the viability of the proposed approach.

Palabras Clave: Nonlinear Output Regulation, Descriptor System, Linear Matrix Inequality, Takagi-Sugeno Model

Abstract

Este artículo presenta una solución numérica al problema de regulación no lineal de salida para sistemas descriptores. La ley de control es no lineal y consiste en un estabilizador combinado con un mapeo lineal en estado estacionario, así como una entrada de estado estacionario no lineal; todos ellos se calculan mediante desigualdades matriciales lineales. Se utilizan un ejemplo numérico y un sistema mecánico para ilustrar la viabilidad del enfoque propuesto.

Keywords: Diseño de regulador no lineal, Sistema descriptor, Desigualdad matricial lineal, Modelo Takagi-Sugeno.

1. Introduction

In control systems, the problem of driving the system output such as it asymptotically tracks a desired reference and rejects undesired disturbances while keeping stability in a closed-loop systems is address frequently due to its wide applicability in mechanical systems, aeronautics and robotics, among others; this task is referred as the output regulation problem (Isidori and Byrnes, 1990). The works of Francis (1977) and Francis and Wonham (1976) have shown that the solvability of a multivariable linear regulator problem corresponds to a system of two linear matrix equations, called Francis Equations. Later, Isidori and Byrnes (1990) shown that the result established by Francis is a particular case of the nonlinear problem, providing necessary and sufficient conditions as a set of nonlinear partial differential equations called Francis-Isidori-Byrnes (FIB). Unfortunately, these conditions presents, in many cases, a considerable numerical complexity. Some numerical solutions have been given, most of them in terms of Takagi-Sugeno (TS) fuzzy models (Chiu and Chiang, 2009; Castillo-Toledo et al., 2012; Chen, 2005; Tapia-Herrera et al., 2013; X.-Jun and Z.-Qi, 2000; Lian and Liou, 2006; Ma and qi Sun, 2000; Lee et al., 2003; Hernández-Cortés et al., 2015; Karamanos et al., 2018) or as dynamic implementations (Armenta et al., 2019). The popularity of TS models is due to their capability to exactly represent (via the sector nonlinearity approach) or approximate with an arbitrary exactness nonlinear dynamics (Tanaka and Sugeno, 1992; Tanaka and Wang, 2001); regardless its origin, a TS model is viewed as a convex combination of linear submodels (vertex models) together with scalar convex functions; such structure facilitates the use the direct Lyapunov's method and obtaining condintions in terms of linear matrix inequalities (LMIs). LMI conditions are highly appreciated since its solvability can be determined by convex optimization tech-

*Autor para correspondencia: tonatiuh@upp.edu.mx
E-mail: luis.poblete@micorreo.upp.edu.mx (Luis Poblete), tonatiuh@upp.edu.mx (Tonatiuh Hernández-Cortés), victor_estrada@upp.edu.mx (V. Estrada-Manzo)

Received: 14/junio/2021 Accepted: 12/08/2021 Published: 05/01/2022
https://doi.org/10.29057/icbi.v9i18.7649

niques (Boyd et al., 1994) already implemented in commercial software (Gahinet et al., 1994; Sturm, 1999). Despite the use of TS models for solving the nonlinear output regulation problem, LMI conditions are only for the stabilizer part of the final control law. Recently in (Bernal et al., 2012b) an LMI solution for the nonlinear mappings is available. Nevertheless, none of the above works deals with nonlinear descriptor systems; they may appear when the Euler-Lagrange formalism is employed for modelling mechanical, biomechanical, electromechanical systems (Lewis et al., 2003; Fantoni et al., 2002; Luenberger, 1979).

This work provides a numerical solution for the output regulation of nonlinear descriptor systems via TS models and LMIs. The proposal extends developments from (Lin and Dai, 1996), where linear singular systems are considered; it also takes advantage of descriptor representations since the input matrix remains constant, and as mentioned in (Meda-Campaña et al., 2009) it relaxes the fuzzy FIB equations under mild assumptions. The designing conditions are in terms of LMIs.

The rest of the work is organized as follows: Section 2 places the problem to be solved and states some standard solutions; Section 3 provides LMI conditions to approximate a solution of the nonlinear output regulation problem for descriptor systems; Section 4 illustrates the performance of the proposal on physical as well as numerical examples. Finally, Section 5 closes the paper and gathers some comments on future work.

2. Problem Statement

Consider the following class of nonlinear descriptor system[1]:

$$E(x)\dot{x}(t) = A(x)x(t) + Bu(t) + D(x)w(t), \quad y(t) = Cx(t), \quad (1)$$

where $x \in \Omega_x \subset \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^o$ is the output vector, and $w \in \Omega_w \subset \mathbb{R}^s$ is the state vector of the exosystem, to be defined later, which generates the reference and/or the perturbation signals; it is assumed that $x = 0$ is an equilibrium point, $A(\cdot), E(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$, $D(\cdot) : \mathbb{R}^n \mapsto \mathbb{R}^s$ are sufficiently smooth for all $x \in \Omega_x$ such that $0 \in \Omega_x$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{o\times n}$ are constant matrices. Matrix $E(\cdot)$ is full rank for $x \in \Omega_x$, i.e., from (1) it is always possible to obtain a standard state-space representation:

$$\dot{x}(t) = f(x, w, u), \quad (2)$$

with $f(x, w, u) = E^{-1}(x)(A(x)x(t) + Bu(t) + D(x)w(t))$. Now let us recall the output regulation problem for system (2) and the following exosystem

$$\dot{w}(t) = s(w), \quad y_r(t) = q(w), \quad (3)$$

where $s(w) : \mathbb{R}^s \mapsto \mathbb{R}^s$ and $q(\cdot) : \mathbb{R}^s \mapsto \mathbb{R}^o$ are sufficiently smooth vector fields holding $s(0) = q(0) = 0$, $y_r \in \mathbb{R}^o$ is called the reference output. The exosystem is poisson stable. Arguments will be omitted when their meaning can be inferred from the context.

In (Isidori and Byrnes, 1990; Isidori, 1995), the nonlinear output regulation problem for systems of the form (2), with a nonlinear output $y(t) = h(x)$, $h(0) = 0$, consists in finding a controller $u = \alpha(x, w)$, $\alpha(0, 0) = 0$ such that

- The equilibrium point $x = 0$ of the closed-loop system $\dot{x} = f(x, 0, \alpha(x, 0))$ is asymptotically stable.
- The tracking error

$$e(t) = y(t) - y_r(t) = h(x) - q(w), \quad (4)$$

goes asymptotically to zero for any initial condition $(x(0), w(0)) \in \Omega \subset \Omega_x \times \Omega_w$.

Then, the following assumptions are considered: A1) the exosystem (3), with $w = 0$ being an equilibrium point, is Poisson stable, that is, the eigenvalues of $S = \partial s(w)/\partial w|_{w=0}$ does not have real part; A2) the linear approximation of (2) at the origin $x = 0$, with $w = 0$, is stabilizable. Thus, the output regulation problem with full information is solvable if and only if there exists mappings $\pi(w) : \mathbb{R}^s \mapsto \mathbb{R}^n$, $\pi(0) = 0$ (the steady-state zero-error manifold) and $\gamma(w) : \mathbb{R}^s \mapsto \mathbb{R}^m$, $\gamma(0) = 0$ (the steady-state input) such that

$$\frac{\partial \pi(w)}{\partial w} s(w) = f(\pi(w), w, \gamma(w)), \quad (5)$$

$$0 = h(\pi(w)) - q(w) \quad (6)$$

holds. The nonlinear control law that performs output regulation is

$$u = K(x - \pi(w)) + \gamma(w) \quad (7)$$

with $K \in \mathbb{R}^{m\times n}$ designed such that the linear approximation of (2), around the origin, is asymptotically stable.

The set of equations (5)-(6) is known as Francis-Isidori-Byrnes (FIB) equations. The next section presents LMI conditions to performing output regulation by means of the original descriptor form (1) and convex representations for a nonlinear stabilization gain and using an adaptation of (5)-(6) to find $\pi(w) = \Pi w$ and $\gamma(w)$.

2.1. Convex representations

There are several methodologies to obtain a Takagi-Sugeno (TS) fuzzy model from (1), among them, the sector nonlinearity approach provides an exact representation (Ohtake et al., 2003). It consists in gathering all the nonlinear terms in $A(x)$, $E(x)$, $D(x)$ in the so-called premise vector $z(x) \in \mathbb{R}^p$such that each entry of $z(x)$ is well-defined and bounded in $\Omega_x \subset \mathbb{R}^n$, i.e., $z_i(x) \in [z_i^0, z_i^1]$, where z_i^0 and z_i^1 are the minimum and maximum of $z_i(x)$. Thus, each term can be expressed as convex sums of its bounds, that is

$$z_i(x) = \mu_0^i(z_i)z_i^0 + \mu_1^i(z_i)z_i^1,$$

[1] In general, descriptor systems are of the form $E(x, w, u, t)\dot{x}(t) = \tilde{f}(x, w, u, t)$ (Duan, 2010). Nevertheless, the class treated in this work (1) naturally appears from state-space representations of systems modeled by Euler-Lagrange formalism $M(q)\ddot{q}(t) + C_o(q, \dot{q})\dot{q}(t) + G(q) = \tau(t)$, $y(t) = q(t)$, where $q \in \mathbb{R}^{n_q}$ is the vector of generalized coordinates, $M(q) \in \mathbb{R}^{n_q\times n_q}$ is the inertia matrix, $C_o(q, \dot{q}) \in \mathbb{R}^{n_q\times n_q}$ is the Coriolis matrix, $G(q) \in \mathbb{R}^{n_q}$ is the gravity vector, and $\tau \in \mathbb{R}^{n_\tau}$ is the torque vector (Lewis et al., 2003, Section 4.3)

where

$$\mu_0^i(z_i) = \frac{z_i^1 - z_i(x)}{z_i^1 - z_i^0}, \quad \mu_1^i(z_i) = 1 - \mu_0^i(z_i),$$

are called weighting functions and fulfill the convex sum property in $\forall x \in \Omega_x$, i.e., $\mu_0^i(z_i) + \mu_1^i(z_i) = 1$ and $\mu_0^i, \mu_1^i \in [0,1]$. Then, the so-called membership functions can be defined as

$$\boldsymbol{\mu}_i(z) = \prod_{j=1}^{p} \mu_{i_j}^p(z_j), \quad i_j = \{0,1\},$$

they also hold the convex sum property: $0 \leq \boldsymbol{\mu}_i(z) \leq 1$ and $\sum_{i=1}^r \boldsymbol{\mu}_i(z) = 1$. Finally, a Takagi-Sugeno model for system (1) can be expressed as follows (Taniguchi et al., 1999):

$$\sum_{i=1}^{r} \boldsymbol{\mu}_i(z) E_i \dot{x} = \sum_{i=1}^{r} \boldsymbol{\mu}_i(z)(A_i x + Bu + D_i w), \quad y = Cx, \quad (8)$$

where $(E_i, A_i, B, C, D_i)|_{\boldsymbol{\mu}_i(z)=1}$ are the vertex matrices, $r = 2^p$ is the number of vertex (rules, linear submodels). In this work, we assume that the exosystem (3) is linear, this is,

$$\dot{w}(t) = S w(t), \quad S \in \mathbb{R}^{s \times s}, \quad (9)$$

with S having its eigenvalues on the imaginary axis. The following section presents an adaptation of the traditional FIB equations for nonlinear descriptor systems in an exact TS fuzzy model (8).

3. Main Results

In (Meda-Campaña et al., 2009), it is established that system (2), with a TS representation, performs output regulation if the input distribution matrix $B(x)$ and the steady-state mapping $\pi(w)$ are constant, while $\gamma(w)$ has a convex form. Based on this and taking advantage of the structure in (8), we define $\pi(w) = \Pi w$, $\Pi \in \mathbb{R}^{n \times s}$ and $\gamma(w) = \sum_{i=1}^r \boldsymbol{\mu}_i(z)\Gamma_i w$, $\Gamma_i \in \mathbb{R}^{m \times s}$, $i \in \{1,2,\ldots,r\}$. Moreover, following the results for singular systems presented by Lin and Dai (1996), we propose the following adapted r-FIB equations for descriptor systems:

$$E_i \Pi S = A_i \Pi + B\Gamma_i + D_i, \quad (10)$$

$$0 = C\Pi - Q. \quad (11)$$

The r set of linear equations must hold simultaneously for Γ_i, $i \in \{1,2,\ldots,r\}$ and a single Π. Note that thanks to the structure of the descriptor systems, there are no crossed-products between Γ_i and B_i (Meda-Campaña et al., 2009). Thus, the nonlinear control law that performs output regulation is

$$u = K_\mu (x - \Pi w) + \Gamma_\mu w \quad (12)$$

with K_μ designed such that the origin of (1) is asymptotically stable.

Remark 1. *The above assumption on matrix B is met when dealing with mechanical systems whose mathematical modeling is obtained from Euler-Lagrange methodology; thus a state-space representation is of the form* (1).

Now, in order to design the stabilizer, based on the descriptor convex model, the augmented system *redundancy form* is used, i.e. $\dot{x} = \dot{x}$ and $0 \times \ddot{x} = A_\mu x + Bu - E_\mu \dot{x}$ as in (Taniguchi et al., 1999, 2000; Arceo et al., 2016), then

$$\bar{E}\dot{\bar{x}} = \bar{A}_\mu \bar{x} + \bar{B}u \quad (13)$$

$$y = \bar{C}\bar{x} \quad (14)$$

where $\bar{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$, $\bar{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{A}_\mu = \begin{bmatrix} 0 & I \\ A_\mu & -E_\mu \end{bmatrix}$, $\bar{B} = \begin{bmatrix} 0 \\ B \end{bmatrix}$, and $\bar{C} = [C \ 0]$. The nonlinear control law under design is a parallel distributed compensation (PDC) one (Wang et al., 1995):

$$u = \sum_{i=1}^{r} \boldsymbol{\mu}_i(z) K_i x = \begin{bmatrix} K_\mu & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \bar{K}_\mu \bar{x}, \quad (15)$$

where $K_i \in \mathbb{R}^{m \times n}$, $i \in \{1,2,\ldots,r\}$ are the controller gains to be designed. To this end consider the following Lyapunov function $V(\bar{x}) = \bar{x}^T \bar{E}^T \bar{P} \bar{x}$, $\bar{E}^T \bar{P} = \bar{P}^T \bar{E} \geq 0$, where$\bar{P} = \begin{bmatrix} P_1 & 0 \\ P_3 & P_4 \end{bmatrix}$, $P_i \in \mathbb{R}^{n \times n}$, $i \in \{1,3,4\}$ such that $P_1 = P_1^T > 0$ and P_4 being invertible. Indeed, guaranteeing $\dot{V} < 0$ ensures the stability of theclosed-loop system; thus we have $\dot{V} < 0$ if

$$\bar{A}_\mu^T \bar{P} + \bar{K}_\mu^T \bar{B}^T \bar{P} + \bar{P}^T \bar{A}_\mu + \bar{P}^T \bar{B} \bar{K}_\mu < 0. \quad (16)$$

Now, considering $X = \begin{bmatrix} X_1 & 0 \\ X_3 & X_4 \end{bmatrix}$, with$X_1 = P_1^{-1}$, $X_4 = P_4^{-1}$, $X_3 = -P_4^{-1} P_3 P_1^{-1}$ and multiplying (16) on the left and right by X^T and X, respectively, yields

$$X^T \bar{A}_\mu^T + \bar{M}_\mu^T \bar{B}^T + \bar{A}_\mu X + \bar{B} \bar{M}_\mu < 0, \quad (17)$$

with $\bar{M}_\mu = \bar{K}_\mu X$. The previous inequality guarantees that $\dot{V} < 0$, thus asymptotic stability at the origin is achieved. Nevertheless, in practical cases it is important to consider the following: guaranteeing a maximum speed convergence $\varphi > 0$ while holding a bound on the control input $\|u(t)\| \leq \beta$, $\beta > 0$. The former is guaranteed if $\dot{V} \leq -2\varphi V$ holds, in terms of LMIs this is implied by

$$\begin{bmatrix} X_3 + X_3^T + 2\varphi X_1 & (*) \\ A_i X_1 + B M_i - E_i X_3 + X_4^T & -E_i X_4 - X_4^T E_i^T \end{bmatrix} < 0, \quad (18)$$

hold for all $i \in \{1,2,\ldots,r\}$. If feasible, the controller gains are computed as $K_i = M_i X_1^{-1}$, $i \in \{1,2,\ldots,r\}$. As for bounding the control law we have

$$\begin{bmatrix} 1 & x^T(0) \\ x(0) & X_1 \end{bmatrix} \geq 0 \text{ and } \begin{bmatrix} X & M_i^T \\ M_i & \beta^2 \end{bmatrix} \geq 0, i \in \{1,2,\ldots,r\}, \quad (19)$$

where $x(0)$ is a given initial condition.

Remark 2. *Based on the proposed results by Bernal et al. (2012a), conditions* (10) *and* (11) *can be written as an LMI problem, thus the design of* Π *and* $\gamma(w(t))$ *can be seen as an optimization problem. Let* $N(x)$ *and* $R(x)$ *be continuously differentiable linear matrix functions and* x *as the decision vector. Then, an approximated solution for* $N(x) - R(x) = 0$ *can be settled as minimization problem, that is,* $\min \epsilon > 0 : -\epsilon \prec N(x) - R(x) \prec \epsilon$, $\prec$ *stands for element-wise ordinary lower than, in other words:*

$$\min \epsilon > 0 : \begin{cases} N(x) - R(x) - \epsilon \prec 0 \\ N(x) - R(x) + \epsilon \succ 0 \end{cases}$$

Thus, by means of Remark 2, conditions (10) and (11) can be expressed as

$$\min \epsilon > 0 :$$
$$-\epsilon \prec \begin{bmatrix} A_i\Pi + B\Gamma_i + D_i - E_i\Pi S & 0 \\ 0 & C\Pi - Q \end{bmatrix} \prec \epsilon \quad (20)$$

with ϵ arbitrarily small.

Remark 3. *The numerical complexity of the LMI approach increases as the number of nonlinearities increases, since the number of vertex models is 2^p. In Xie et al. (2014) the numerical complexity of the LMI conditions is approximated by $\log_{10}(n_d^3 n_l)$, where n_d is the number of scalar decision variables and n_l is the number of LMI rows. In our case, as the complete design involves LMIs* (18), (19) *and* (20) *we have $n_d = 0.5n(n+1) + 2n^2 + mnr + ns + msr + 1$ and $n_l = 2nr + n + 1 + r(n+m) + 2(n+s)r + 2(o+s)$, where n is the number of states, m the number of inputs, o the number of outputs, s the size of the exosystem, $r = 2^p$ is the number of vertex models.*

4. Examples

In this section, the proposed fuzzy regulation approach and method previously derived are applied to two TS descriptor fuzzy models without external disturbances. The first example is numerical while the second on corresponds to the well-known cart-pole system. The LMI conditions have been checked by the LMItoolbox (Gahinet et al., 1994) within Matlab 2018a.

4.1. Numerical example

Consider a nonlinear model of the form (1) with

$$E(x) = \begin{bmatrix} 0.37 & 0.43 - 2\frac{1}{1+x_2^2} \\ 0.23 & 1.15 \end{bmatrix}, A(x) = \begin{bmatrix} -0.35 & 1.24 + \cos x_2 \\ -0.74 & 0.5 \end{bmatrix},$$
$$B = \begin{bmatrix} 1.5 \\ 0 \end{bmatrix}, D(x) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 \\ 1 \end{bmatrix}^T.$$

Applying the sector no linearity approach stated in Section 2.1; two different nonlinear terms can be identified $z_1 = 1/(1 + x_2^2) \in [0,\ 1]$ and $z_2 = \cos x_2 \in [-1,\ 1]$, such bounds hold in $\Omega_x = \mathbb{R}^2$. Then, the weighting functions are $\mu_0^1 = (1 - z_1)$, $\mu_1^1 = 1 - \mu_0^1$, $\mu_0^2 = 0.5(1 - z_2)$, $\mu_1^2 = 1 - \mu_0^2$; therefore, the four membership functions are $\boldsymbol{\mu}_1 = \mu_0^1\mu_0^2$, $\boldsymbol{\mu}_2 = \mu_0^1\mu_1^2$, $\boldsymbol{\mu}_3 = \mu_1^1\mu_0^2$, and $\boldsymbol{\mu}_4 = \mu_1^1\mu_1^2$. The vertex matrices are

$$E_1 = E_2 = \begin{bmatrix} 0.37 & 0.43 \\ 0.23 & 1.15 \end{bmatrix}, E_3 = E_4 = \begin{bmatrix} 0.37 & -1.57 \\ 0.23 & 1.15 \end{bmatrix},$$

$$A_1 = A_3 = \begin{bmatrix} -0.35 & 0.24 \\ -0.74 & 0.5 \end{bmatrix}, A_2 = A_4 = \begin{bmatrix} -0.35 & 2.24 \\ -0.74 & 0.5 \end{bmatrix}.$$

The considered exosystem is (9) with $\dot{w}_1 = w_2$, $\dot{w}_2 = -w_1$. The task is that x_2 tracks $w_1 + w_2$, thus $Q = [1\ 1]$. Running LMIs (18) with $\phi = 0$ (no decay rate condition) together with (20), feasible solutions are found with $\epsilon = 9.0757 \times 10^{-16}$, meaning that the FIB equations (10)-(11) are satisfied. The computed gains are

$$\Pi = \begin{bmatrix} 1.784 & -1.433 \\ 1 & 1 \end{bmatrix}, \Gamma_1 = \begin{bmatrix} 0.323 \\ 0.232 \end{bmatrix}^T, \Gamma_2 = \begin{bmatrix} -1.010 \\ -1.100 \end{bmatrix}^T,$$
$$\Gamma_3 = \begin{bmatrix} 1.656 \\ -1.100 \end{bmatrix}^T, \Gamma_4 = \begin{bmatrix} 0.323 \\ -2.434 \end{bmatrix}^T, K_1 = \begin{bmatrix} -0.773 \\ 1.386 \end{bmatrix}^T,$$
$$K_2 = \begin{bmatrix} -0.773 \\ 0.053 \end{bmatrix}^T, K_3 = \begin{bmatrix} -0.151 \\ 1.172 \end{bmatrix}^T, \text{ and } K_4 = \begin{bmatrix} -0.151 \\ -0.160 \end{bmatrix}^T.$$

Thus, now we are ready to implement the control law (12) for the nonlinear system. Simulation results for initial conditions $x(0) = [1\ 1.5]^T$ and $w(0) = [0.7\ 0]^T$ are shown in Figures 1 and 2; it can be seen that the tracking takes place asymptotically.

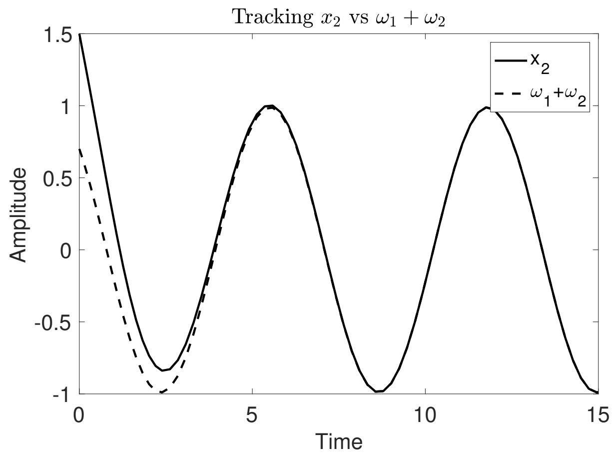

Figure 1: Output versus reference x_2 vs $w_1 + w_2$

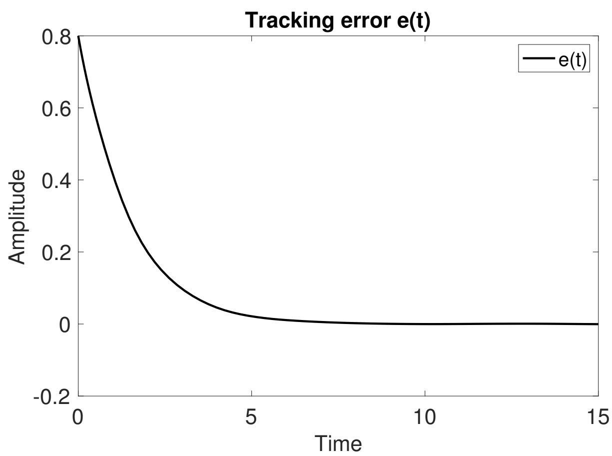

Figure 2: control signal $u(t)$ of the system

It is important to notice that the system under study has a linear input distribution matrix and if a standard state-space representation (2) is to be obtained, matrix B would be no longer constant, thus results reported in (Meda-Campaña et al., 2009) cannot be directly applied. Additionally, inverting the matrix $E(x)$ would generate more nonlinear terms and therefore more vertex (rules) models.

4.2. Cart-pole system

Consider the underactuated system in the Figure 3 consisting for a car on the rail and one vertical beam joined to the car (Fantoni et al., 2002). The cart-pole dynamical equations can be represented from the Lagrange equation of motion (Lewis et al., 2003), whose final form is $M(q)\ddot{q} + C_o(q,\dot{q})\dot{q} + G(q) = \tau$, where $q = [x \ \theta]^T$ is a vector of generalized coordinates x is the distance of the horizontal rail and θ is the angle from the vertical, i.e.

$$M(q) = \begin{bmatrix} M+m & ml\cos\theta \\ ml\cos\theta & ml^2 \end{bmatrix}, \ G(q) = \begin{bmatrix} 0 \\ -mgl\sin\theta \end{bmatrix}$$

$$C_o(q,\dot{q}) = \begin{bmatrix} 0 & -ml\sin\theta\dot{\theta} \\ 0 & 0 \end{bmatrix}, \text{ and } \tau = \begin{bmatrix} f \\ 0 \end{bmatrix}.$$

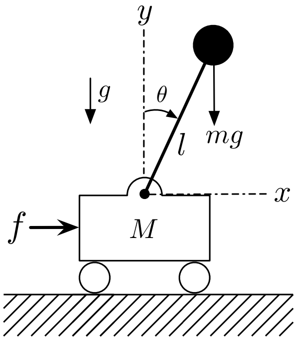

Figure 3: Cart-Pole System

Table 1: Nonlinearities in (21) and their bounds

Nonlinearity	Definition	Lower bounds z_i^0	Upper bounds z_i^1
z_1	$\sin x_3$	-0.2588	0.2582
z_2	$\sin x_3/x_3$	0.9886	1
z_3	x_4	-1.5708	1.5702
z_4	$\cos x_3$	0.9659	1

The motion equation for this system, described by its descriptor model (1) is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & M+m & 0 & ml\cos x_3 \\ 0 & 0 & 1 & 0 \\ 0 & ml\cos x_3 & 0 & ml^2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ ml\sin x_3 x_4^2 + u \\ x_4 \\ mlg\sin x_3 \end{bmatrix}. \quad (21)$$

Rewriting the previous nonlinear descriptor as a convex sum in a compact set $\Omega_x = \{|x_3| \leq 15°, |x_4| \leq 1\text{rad/s}\}$ with their nonlinearities and their bounds settle in the table 1, we have an exact TS representation:

$$\sum_{i=1}^{16} \boldsymbol{\mu}_i(z) E_i \dot{x} = \sum_{i=1}^{16} \boldsymbol{\mu}_i(z)(A_i x + Bu),$$

where $\boldsymbol{\mu}_i(z) = \mu_{i_1}^1(z)\mu_{i_2}^2(z)\mu_{i_3}^3(z)\mu_{i_4}^4(z)$ and the indexes $[i_1 i_2 i_3 i_4]$ are the digit binary representation of $(i-1)$, $i \in \{1, 2, \ldots, 16\}$. A set of illustration matrices A_i, E_i and $B = [0 \ 1 \ 0 \ 0]^T$ for the cart-pole system are given below:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.024 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0.582 & 0 \end{bmatrix}, A_8 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.024 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0.588 & 0 \end{bmatrix},$$

$$A_{13} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.024 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0.588 & 0 \end{bmatrix}, A_{16} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0.024 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0.588 & 0 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1.6 & 0 & 0.058 \\ 0 & 0 & 1 & 0 \\ 0 & 0.058 & 0 & 0.009 \end{bmatrix}, E_8 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1.6 & 0 & 0.058 \\ 0 & 0 & 1 & 0 \\ 0 & 0.058 & 0 & 0.009 \end{bmatrix},$$

$$E_{13} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1.6 & 0 & 0.060 \\ 0 & 0 & 1 & 0 \\ 0 & 0.060 & 0 & 0.009 \end{bmatrix}, E_{16} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1.6 & 0 & 0.060 \\ 0 & 0 & 1 & 0 \\ 0 & 0.060 & 0 & 0.009 \end{bmatrix}.$$

For this system, the task is that pendulum follows a sinusoidal reference, that is, x_3 asymptotically tracks ω_1, generated by the exosystem $\dot{\omega}_1 = \omega_2$, $\dot{\omega}_2 = -\omega_1$. As mentioned before by using the LMIs (18) with $\varphi = 0$ and (20), the controller (12) is computed by the established in Section 3. The LMIs are found feasible, for illustration purposes some of the gains are shown

$$\begin{aligned} K_1 &= [446.0 \quad 435.4 \quad 1428.5 \quad 236.8], \\ K_8 &= [446.2 \quad 435.6 \quad 1429.1 \quad 237.0], \\ K_{13} &= [446.0 \quad 435.4 \quad 1428.5 \quad 236.9], \\ K_{16} &= [446.2 \quad 435.6 \quad 1429.1 \quad 236.9]; \end{aligned}$$

the minimum ϵ is 0.0135, the common steady state

$$\Pi = \begin{bmatrix} -9.9388 & 0 \\ 0 & -9.9388 \\ 0.9865 & 0 \\ 0 & 0.9773 \end{bmatrix},$$

and some of steady state inputs Γ_i related to i subsystems $\Gamma_1 = \begin{bmatrix} 15.845 & -0.023 \end{bmatrix}$, $\Gamma_8 = \begin{bmatrix} 15.843 & 0.023 \end{bmatrix}$, $\Gamma_{13} = \begin{bmatrix} 15.845 & 0.023 \end{bmatrix}$, $\Gamma_{16} = \begin{bmatrix} 15.843 & -0.023 \end{bmatrix}$. Simulation results have been performed for initial conditions $x(0) = [10° \ 0.05 \ 0.1 \ 0.05]^T$, $w(0) = [5 \ 0]^T$, the behavior closed-loop trajectories are depicted in Figure 4; it can be observed, the tracking error goes to zero as time increases. The rest of the states and the control input are shown in Figure 5, notice that at the beginning, the control law requires a large amount of energy.

Keep in mind that as we are in the LMI framework, we can directly add performance specifications such that decay rate, input-output constraints, etc. Since the control law would be applied to the real system, lower input control signal is required and the necessity of bounds in order to not affect the actuator. Such improvements can performed by running LMIs (18) for decay rate $\varphi = 0.8$ and (19) for the input constrain $\beta = 10$ with $x(0) = [0 \quad 0 \quad 5° \quad 0]^T$ as initial condition. As expected, the control signal holds the imposed bounds and now it can be implemented directly on the physical system; its performance can be seen in Figures 6 and 7.

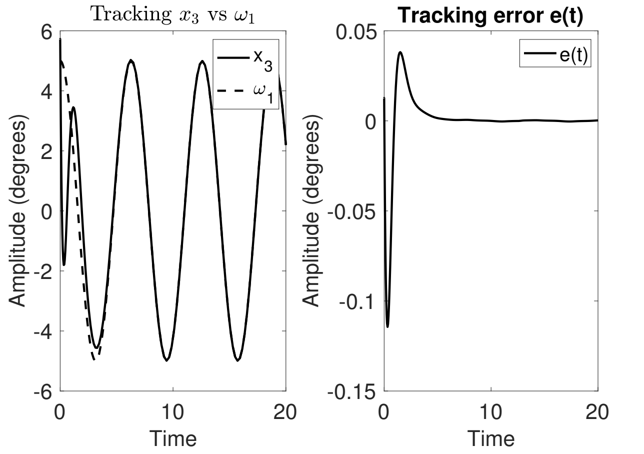

Figure 4: Output x_3 versus reference ω_1 and tracking error $e(t)$ for the cart-pole

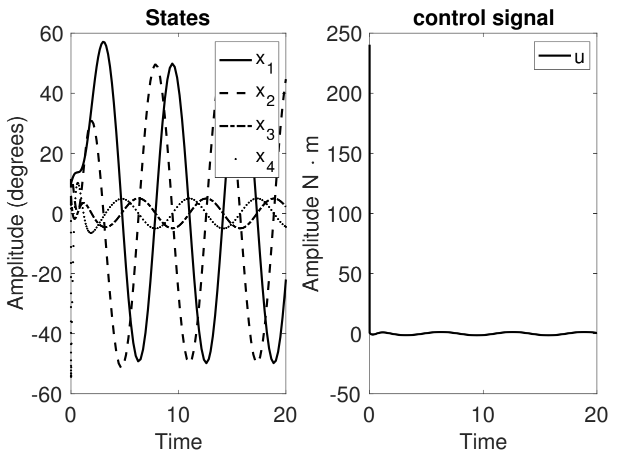

Figure 5: States and control signal $u(t)$ for the nonlinear cart-pole system

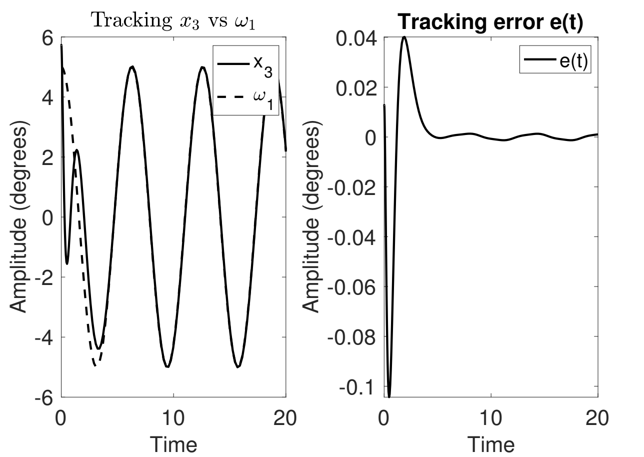

Figure 6: Output x_3 versus reference ω_1 and tracking error $e(t)$ for the cart-pole system with decay rate and input constraints

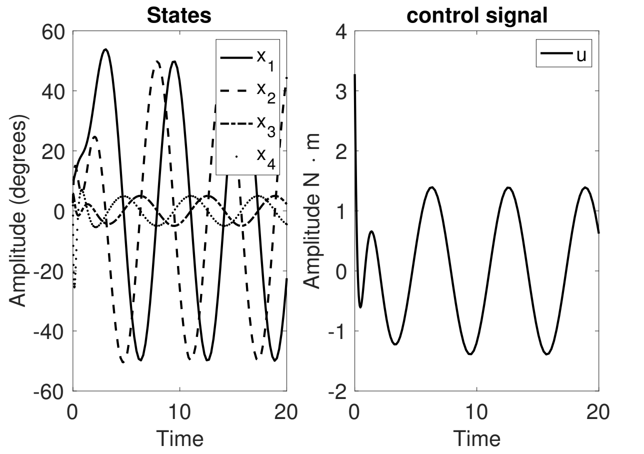

Figure 7: States and control signal $u(t)$ for the nonlinear cart-pole system with decay rate and input constraints

5. Conclusions

This work has defined the nonlinear output regulation for dynamical systems in descriptor forms; sufficient conditions for the existence of such controller are given in terms of linear matrix inequalities. In particular, the proposed approach are given by two components, the first one is a nonlinear stabilizer which creates a globally attractive steady state and, the second one, the steady-state mappings on the basis on new equations in the sense of Isidori; all of these carried out in a practical way. It has been shown that, the approach can be effective for the implementation in real time. Two systems have been presented in order to show the advantages of the proposed approach.

Acknowledgments

This work has been support via the CONACYT scholarship for CVU 585090, the laboratory LANAVEX of the Universidad Politécnica de Pachuca.

References

Arceo, J. C., Vázquez, D., Estrada-Manzo, V., Márquez, R., Bernal, M., 2016. Nonlinear convex control of the furuta pendulum based on its descriptor model. In: 2016 13th International Conference on Electrical Engineering, Computing Science and Automatic Control (CCE). IEEE, pp. 1–6.

Armenta, C., Álvarez, J., Márquez, R., Bernal, M., 2019. A practical solution to implement nonlinear output regulation via dynamic mappings. Kybernetika 55 (2), 385–401.

Bernal, M., Marquez, R., Estrada-Manzo, V., Castillo-Toledo, B., 2012a. An element-wise linear matrix inequality approach for output regulation problems. In: World Automation Congress 2012. IEEE, pp. 1–6.

Bernal, M., Marquez, R., Estrada-Manzo, V., Castillo-Toledo, B., 2012b. Nonlinear output regulation via Takagi-Sugeno fuzzy mappings: A full-information LMI approach. In: 2012 IEEE International Conference on Fuzzy Systems. IEEE, pp. 1–7.

Boyd, S., Ghaoui, L. E., Feron, E., Balakrishnan, V., 1994. Linear Matrix Inequalities in Systems and Control Theory. SIAM, Philadelphia, PA.

Castillo-Toledo, B., Di Gennaro, S., Jurado, F., June 2012. Trajectory tracking for a quadrotor via fuzzy regulation. In: World Automation Congress (WAC), 2012. pp. 1–6.

Chen, S., 2005. Output regulation of nonlinear singularly perturbed systems based on t-s fuzzy model. Journal of Control Theory and Applications 3 (4), 399–403.

Chiu, C.-S., Chiang, T.-S., Aug 2009. Robust output regulation of t–s fuzzy systems with multiple time-varying state and input delays. Fuzzy Systems, IEEE Transactions on 17 (4), 962–975.
DOI: 10.1109/TFUZZ.2009.2017013

Duan, G.-R., 2010. Analysis and design of descriptor linear systems. Vol. 23. Springer Science & Business Media.

Fantoni, I., Lozano, R., Sinha, S., 2002. Non-linear control for underactuated mechanical systems. Appl. Mech. Rev. 55 (4), B67–B68.

Francis, B. A., 1977. The linear multivariable regulator problem. SIAM J. Cont. Optimiz. 15, 486–505.

Francis, B. A., Wonham, W. M., 1976. The internal model principle of control theory. Automatica 12, 457–465.

Gahinet, P., Nemirovskii, A., Laub, A. J., Chilali, M., 1994. The LMI control toolbox. In: Proceedings of 1994 33rd IEEE Conference on Decision and Control. Vol. 3. IEEE, pp. 2038–2041.

Hernández-Cortés, T., Meda Campaña, J. A., Páramo Carranza, L. A., Gómez Mancilla, J. C., 2015. A simplified output regulator for a class of takagi-sugeno fuzzy models. Mathematical Problems in Engineering.

Isidori, A., 1995. Nonlinear Control Systems. Springer, Berlin.

Isidori, A., Byrnes, C. I., 1990. Output regulation of nonlinear systems. IEEE Transactions on Automatic Control 35 (2), 131–140.

Karamanos, K., Aguilar-Ibanez, C., Meda-Campana, J. A., Suarez-Castanon, M. S., Rubio, J. d. J., Cruz-Cortes, N., 2018. On the output regulation problem: The generalized second-order underactuated linear system case. Mathematical Problems in Engineering 2018, 3820935.
URL: https://doi.org/10.1155/2018/3820935
DOI: 10.1155/2018/3820935

Lee, H. J., Park, J. B., Joo, Y. H., June 2003. Comments on output tracking and regulation on nonlinear system based on Takagi-Sugeno fuzzy model. IEEE Transactions on Systems, Man, and Cybertics-Part B:Cybernetics 33 (3), 521–523.

Lewis, F. L., Dawson, D. M., Abdallah, C. T., 2003. Robot manipulator control: theory and practice. CRC Press.

Lian, K.-Y., Liou, J.-J., 2006. Output tracking control for fuzzy systems via output feedback design. Fuzzy Systems, IEEE Transactions on 14 (5), 628–639.
DOI: 10.1109/TFUZZ.2006.876725

Lin, W., Dai, L., 1996. Solutions to the output regulation problem of linear singular systems. Automatica 32 (12), 1713–1718.

Luenberger, D. G., 1979. Non-linear descriptor systems. Journal of Economic Dynamics and Control 1 (3), 219–242.

Ma, X.-J., qi Sun, Z., Feb 2000. Output tracking and regulation of nonlinear system based on takagi-sugeno fuzzy model. Systems, Man, and Cybernetics, Part B: Cybernetics, IEEE Transactions on 30 (1), 47–59.
DOI: 10.1109/3477.826946

Meda-Campaña, J. A., Castillo-Toledo, B., Chen, G., 2009. Synchronization of chaotic systems from a fuzzy regulation approach. Fuzzy Sets and Systems 160.

Ohtake, H., Tanaka, K., Wang, H. O., 2003. Fuzzy modeling via sector nonlinearity concept. Integrated Computer-Aided Engineering 10 (4), 333–341.

Sturm, J. F., 1999. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. Optimization methods and software 11 (1-4), 625–653.

Tanaka, K., Sugeno, M., 1992. Stability analysis and design of fuzzy control systems. Fuzzy sets Syst. 45 (2), 135–156.

Tanaka, K., Wang, H. O., 2001. Fuzzy Control Systems Design and Analysis. A Linear Matrix Inequality Approach. John Wiley and Sons, Inc.

Taniguchi, T., Tanaka, K., Wang, H. O., 2000. Fuzzy descriptor systems and nonlinear model following control. IEEE Transactions on Fuzzy Systems 8 (4), 442–452.
DOI: 10.1109/91.868950

Taniguchi, T., Tanaka, K., Yamafuji, K., Wang, H. O., 1999. Fuzzy descriptor systems: stability analysis and design via lmis. In: Proceedings of the 1999 American Control Conference (Cat. No. 99CH36251). Vol. 3. pp. 1827–1831 vol.3.
DOI: 10.1109/ACC.1999.786165

Tapia-Herrera, R., Meda-Campaña, J. A., Alcántara-Montes, S., Hernández-Cortés, T., 2013. Tuning of a TS fuzzy output regulator using the steepest descent aproach and anfis. Mathematical Problems in Engineering 2013, 14.

Wang, H. O., Tanaka, K., Griffin, M., 1995. Parallel distributed compensation of nonlinear systems by takagi-sugeno fuzzy model. In: Proceedings of 1995 IEEE International Conference on Fuzzy Systems. Vol. 2. IEEE, pp. 531–538.

X.-Jun, M., Z.-Qi, S., February 2000. Output tracking and regulation on nonlinear system based on Takagi-Sugeno fuzzy model. IEEE Transactions on Systems, Man, and Cybertics-Part B:Cybernetics 30 (1), 47–59.

Xie, X., Yue, D., Ma, T., Zhu, X., 2014. Further studies on control synthesis of discrete-time TS fuzzy systems via augmented multi-indexed matrix approach. IEEE Transactions on Cybernetics 44 (12), 2784–2791.