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A new method for the characterization of the dynamics of coupled Hénon map lattices
Un nuevo método de caracterización de la dinámica de redes de mapas de Hénon

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Abstract

A method is proposed to characterize spatially extended non-linear dynamic systems that exhibit both periodic and chaotic spatiotemporal behavior. The system is a two-dimensional square lattice of coupled Hénon maps that interact with nearest neighbors through diffusive coupling. Focusing on just one of the maps of the lattice (network) and on one of its dynamic variables, this method is compared against two other forms of characterization of the dynamic behavior of the network. It is shown that the sampling of a single map provides more consistent and effective results than the other two methods.

Keywords: Hénon map, Coupled Map Lattices, Diffusive coupling, dynamics characterization.

Resumen

Se analiza un método para caracterizar sistemas dinámicos no lineales espacialmente extendidos que exhiben un comportamiento espaciotemporal tanto periódico como caótico. El sistema analizado es una red cuadrada bidimensional de mapas de Hénon acoplados que interactúan con los vecinos más cercanos a través de un acoplamiento difusivo. Enfocándose en uno de los mapas de la red y midiendo una de sus variables dinámicas, se compara este método contra otras dos formas de caracterización del comportamiento dinámico de la red. Se demuestra que el muestreo de un solo mapa proporciona resultados más consistentes y efectivos que los otros dos métodos.

Palabras Clave: Mapa de Hénon, Arreglos de mapas acoplados, Acoplamiento difusivo, Caracterización dinámica.

1. Introduction

The significance of Coupled Map Lattices (CML) (Kaneko, 1992), (Kaneko, 1991) is based on the fact that they are useful for developing and understanding new concepts in spatiotemporal systems as well as deducing universal laws (Zhang and Wang, 2015), (Yang et al., 1996), (Liu et al., 1999). A critical not completely understood issue of nonlinear dynamical systems is to determine the way in which a periodic or chaotic behavior influences the dynamics of these systems with a large number of degrees of freedom such as coupled systems (Tran, 2001), (Chakravarty et al., 2003), (Wang et al., 2011). There are a range of applications in which pattern formation and spatiotemporal chaotic behavior (Xu et al., 2019), (Turing, 1952) takes place, including excitable media (Báscones et al., 2002), (Winfree, 1991), (Pande and Pandit, 2000), biological systems (Ahmed et al., 2001), (Nicolis et al., 2004), diffusion fields (Daccord et al., 1986), (Garik et al., 1989), (Meixner et al., 2000), convection (Chiam et al., 2003), (Zhang and Viñals, 1995), and chemical systems (Xi et al., 1993), (Elezgaray and Arneodo, 1992), (O'Hern et al., 1996). Just few analytical results for special nontrivial models which cannot be studied by existing elementary methods have been obtained (Just, 1998), (Lü and Hu, 2004), so to analyze these kinds of complex problems it is useful to continue with studies based on numerical calculations. This work intends to cover this issue by analyzing a method to characterize the temporal dynamics of CMLs.

Hénon map has been used in recent years to understand and extend concepts in nonlinear dynamics (Hu et al., 2017); it is used too in applications in several areas such as parameter fitting (Tao et al., 2004), (Oprisan, 2002), chaos control (Wagner and Stoop, 2001), and betatron oscillations (Tzenov and Davidson, 2003) among others that can be found in literature. Numerical solutions of this map give a rich variety of regular periodic solutions and deterministic chaos; in this

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way, the Hénon map is already a standard tool to analyze nonlinear dynamical systems, here it is utilized to characterize a lattice system.

In characterizing dynamical systems several topics must be considered, some methods require a precise and unbiased characterization of the available information that can be derived from the system such as maximum entropy methods. Also, the idea of information content has found key applications in systems that contain large amounts of information as well as a field to apply formal mathematics. Information content provides a rigorous definition of randomness and a quantitatively precise way of characterizing a particular dynamical system as complex.

To characterize a spatially extended non-linear dynamical system that exhibit spatiotemporal chaotic behavior it has been used several methods as in (Cross and Hohenberg, 1993), (Qi et al., 2003), (Sharma and Gupte, 2002). For CMLs, the exploration of the parameter space is computationally demanding so it is necessary to use a simple characteristic that it be easy to evaluate in order to characterize the states of the system.

There is a diversity of techniques for characterizing a CML, two of the most used are the calculation of the trace of the matrix of stationary states (Just, 1998) and the Lyapunov exponent (Lai et al., 2003). In this work, both techniques are compared against a method proposed by obtaining the sampling of the fast variable of the Hénon map from a single randomly chosen node in the central region of the square network. All these three methods are non-heavy numerical evaluation and are carried out along with the evolution of the system.

This paper is organized as follows. Section 2 presents the assumptions and equations of the model. Section 3 contains the main numerical results and discussions. Finally, section 4 gives the general conclusions.

2. The model

The system under analysis is a discrete two dimensional (2D) squared lattice of Hénon maps defined by equations (1-4). Although the local dynamics of each site in the lattice is described by a Hénon map, here the main focus of attention is on how the spatiotemporal dynamics of the lattice evolves as a whole, where each map is interacting with its nearest-neighbors via Turing's diffusive coupling (Kaneko, 1992). The whole model is defined by

$$x_{i,j}(t+1) = (1-\eta) f_1(x_{i,j}(t), y_{i,j}(t)) + \frac{\eta}{4}\left[\sum_{nn} f_1(x_{i,j}(t), y_{i,j}(t))\right] \quad (1)$$

$$y_{i,j}(t+1) = (1-\eta) f_2(x_{i,j}(t), y_{i,j}(t)) + \frac{\eta}{4}\left[\sum_{nn} f_2(x_{i,j}(t), y_{i,j}(t))\right] \quad (2)$$

where the so called fast and slow variables of the map are respectively

$$x_{i,j}(t) \equiv f_1(x_{i,j}(t), y_{i,j}(t)) = \mu - x_{i,j}^2(t) + J y_{i,j}(t) \quad (3)$$

$$y_{i,j}(t) \equiv f_2(x_{i,j}(t), y_{i,j}(t)) = x_{i,j}(t) \quad (4)$$

and the pair (i, j) labels the row and the column of the matrix that represents the lattice array. Parameters values for μ and J are chosen so that a single Hénon map exhibits both periodic and chaotic dynamic (from here, J is fixed to *0.30*). The coupling strength is carried out by doing summations of four nearest neighbors *(nn)* and scaled with the coupling constant η allowed to vary in the range *(0.0, 0.40)*. The size of the lattice is *100x100* sites with periodic boundary conditions imposed. This fixed medium size of the lattice using diffusive coupling with maps was chosen because it is known from previous test calculations this size and above, the lattice size does not influence the dynamic state as it is reported in (Kaneko, 1991) with a Logistic CML.

Proposed method steps consist first, in assigning a random number to dynamical variables of whole network and evolving in time by iterating equations (1-4). Second, nearest-neighbors of each map interact through constant linear diffusive coupling defined by equations (1) and (2). Third, a node of the lattice is randomly chosen and periodic conditions were imposed.

Selected values of μ were in the range *(0, 2.0)*, for which a solution of one Hénon map goes through periodic to windows of chaotic states. For values outside these ranges of parameter μ, the solution for a single map diverges.

Numerical calculations were performed using a program encoded in C++ language developed by one of the authors (JMSS). For the display of images in Figure 1, the Matlab™ software was used. The display of results in graphs of coordinate systems (figures 2 and 3) was carried out using the Origin™ software.

3. Numerical Results

It was found that once a steady state is reached (after about 200 time-iterations), the lattice and individually each site remains in the same state, i.e. periodic or chaotic. In the case of periodic states, each element of the lattice visits the corresponding values associated to the given state so, for period-1 state has one allowable value, period-2 has two values, and so on. By mapping each value to a gray level and displaying it as an image, a spatial distribution or pattern is form as shown in Figure 1 with $\eta=0.01$ where has been used several values of parameter μ. While the system remains in a low order period, such as period-one or period-two, its dynamic is easily identified through the formed patterns, as in figures 1a, 1b and 1c, but when the system is in a higher periodic order as period eight as in figure 1e, or in a chaotic state as in figures 1d and 1f, the spatial distribution appears to be a random image, therefore cannot be distinguished any state even qualitatively.

With the aim to compare, calculations for some values of the diffusive coupling parameter η were done in terms of the parameter μ. Figure 2 shows the results of the bifurcation diagrams for two methods of identification, the left-hand side column corresponds to the sampling of one point of the lattice proposed here, while the right-hand side column corresponds to the trace of the whole lattice matrix. By analyzing the bifurcation diagrams, it can be seen that, except for period-1 ($\mu<0.40$) for the trace of the matrix calculation the qualitative behavior is non uniform for increasing values of the diffusive coupling constant, even for the trivial case $\eta=0.0$, while for the one sample method (OSM) results to be uniform as the parameter μ is increasing. For further values of η none characterization of this dynamical system is possible to get if it is based just on this technique; moreover, for each value of η the appearance of the bifurcation diagram changes in behavior almost completely, so this technique is inappropriate to characterize Hénon Coupled Map Lattices.

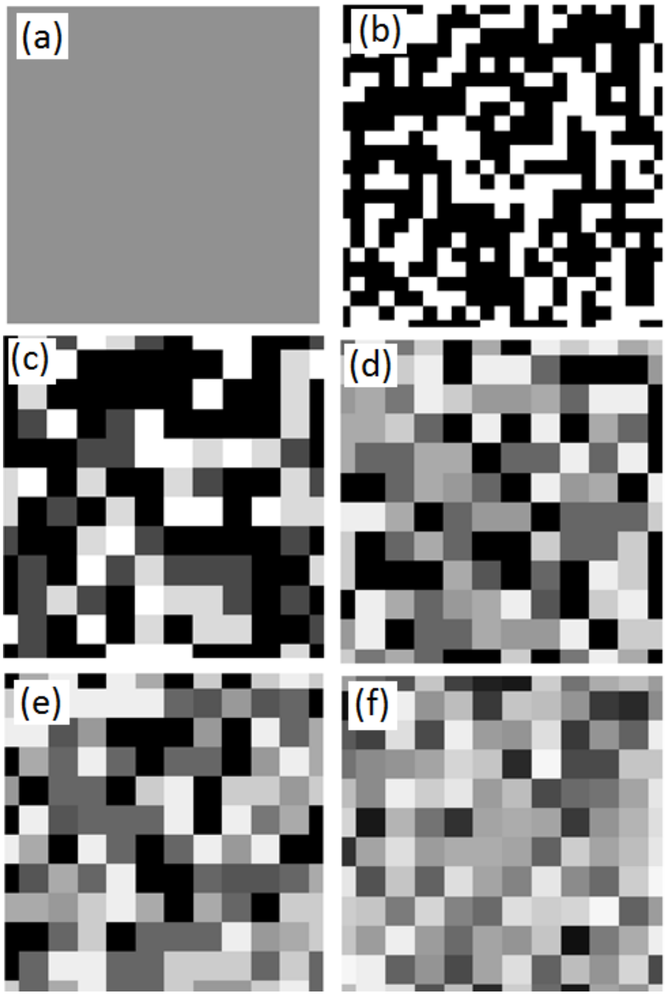

Figure 1: Spatial patterns obtained for several values of the μ-parameter and using a constant diffusive coupling $\eta=0.01$. a) Period one ($\mu=0.20$); b) Period two ($\mu=0.80$); c) Period four ($\mu=1.04$); d) Chaotic state ($\mu=1.17$); e) Period eight window ($\mu=1.27$); f) Chaotic state ($\mu=1.40$). It is shown a sub region of the whole considered lattice in each case.

To obtain first row of Figure 2 a null coupling was applied (n=o) which is the trivial case of isolated maps, therefore, the bifurcation diagram is just the same as for one Hénon map with the same parameters.

As the value of the diffusive coupling constant is increasing, the bifurcation diagram is evolving accordingly. In the range $\mu<0.4$, the trace-matrix method characterizes the system exactly for all values of the coupling constant, however for values $\mu>0.4$, this method fails and the state of the system cannot be determined.

Third method, Lyapunov exponents were obtained to compare with the method proposed here. For $\eta=0.01$, the description given by curve in Figure 3a agrees exactly with the behavior for our method shown in Figure 2 for the same value of η, resulting identical dynamics states in the full range of μ. Although the calculation of the Lyapunov exponent is a very reliable technique for characterizing a dynamic system, in the case of the CML analyzed here, this calculation loses structure for larger values of η ($\eta>0.01$). That happens when the system goes into a quasi-periodic regime ($\mu>0.37$) see Figure 3b. The exponent results positive in this interval, so no conclusion can be made about the dynamics of the lattice, this does not happen in the same range of μ parameter for the case $\eta=0.05$ of the OSM. By comparing both graphs, the calculation of Lyapunov exponents results in an instability (Figure 3b) while the OSM (Figure 2) shows truly the quasi-periodic regime instead.

After the contrast between method proposed in this work versus other two techniques, it can be observed a robust characterization of the system for the entire ranges of values of the parameter μ by the OSM proposed in this work.

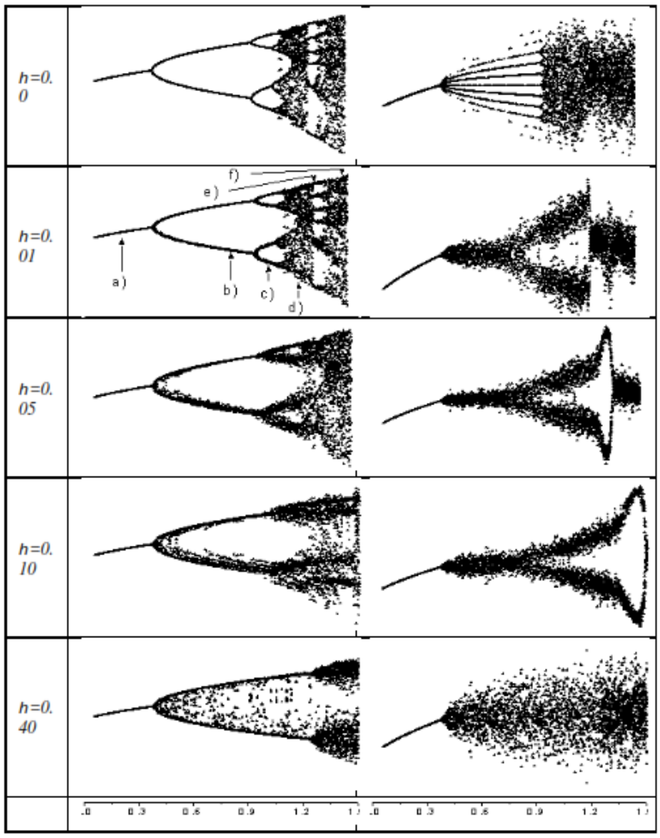

Figure 2: Comparison between two methods of characterization through bifurcation diagrams. First column contains the results for OSM and second column for the trace of the matrix. The horizontal axis corresponds to the parameter μ, the vertical axis (not showed) is in arbitrary units. Arrows in the left hand figure for row with $\eta=0.01$ correspond to each pattern of Figure 1.

4. Conclusions

Exploring the parameter space of a CML is computationally demanding, so it is necessary to use less processor-intensive calculations to characterize the dynamics regimes of the system. Whenever the system be in a low periodic order its dynamic is easily identified through the formed spatial patterns, but when the system be in a higher periodic order or in a chaotic state, the spatial distributions appear to be as a random image in the depiction used here.

The lack of robustness in calculating the trace of the matrix for a lattice of Hénon maps became evident and moreover, the behavior of the bifurcation diagrams for increasing values of the diffusive coupling constant follows a non-uniform behavior even for the trivial case ($\eta=0.0$), no match exists with the isolated Hénon map except in the period-1 region. In the period-2 regime the trace method is ineffective and does not give an intelligible pattern. Resuming, both the trace matrix and Lyapunov exponent method do not characterize the actual cascade of instabilities. Instead, the bifurcation diagram corresponding to one sample of the method proposed here shows a very definite dynamical state.

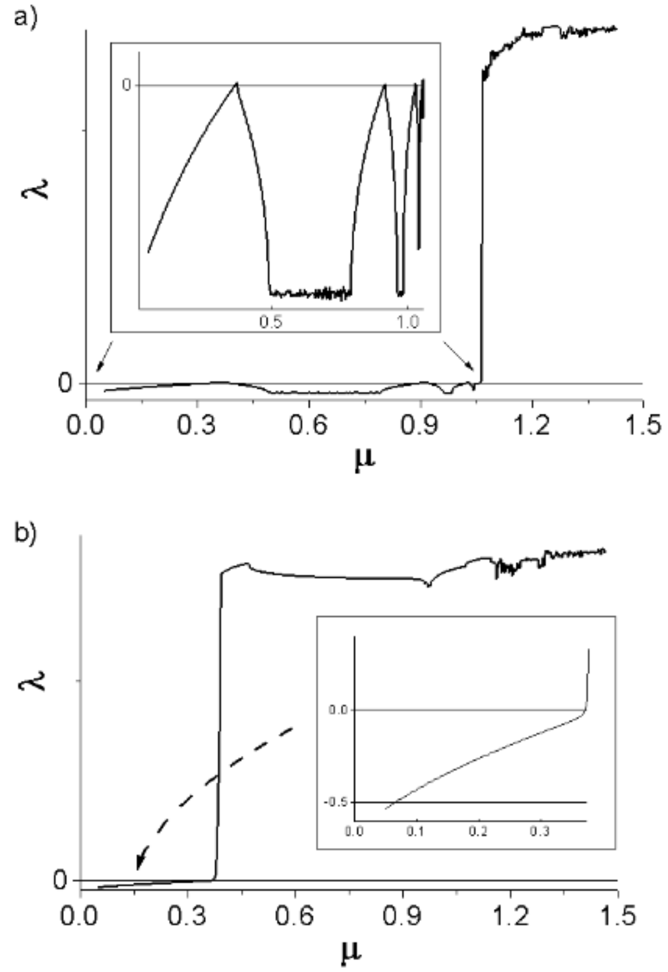

Figure 3: (a) Lyapunov exponents for $\eta=0.01$, and (b) $\eta=0.05$. Once the exponent becomes positive only an instability can be inferred in the ranges $\mu>1.10$ in (a) and $\mu>0.40$ in (b). The vertical axis is in arbitrary units.

By building bifurcation diagrams respect of the parameter μ, it becomes clear that to characterize a CML, it is more efficient to use one site of the lattice instead of a global character of the whole system. As the value of the diffusive coupling constant is increasing, the bifurcation diagram of one lattice point suffers continuous transformation, it has been found that this global characteristic is insensitive to the dynamics of the global lattice.

In the case of the CML, analyzed for increasing values of η, Lyapunov exponent lose structure and just an instability region can be inferred from its graph while the method proposed truly shows the quasi-periodicity regime for such parameter values. So, the characterization of the dynamics of a CML through bifurcation diagrams in the region of stable periodic, cuasi-periodic, and chaotic regime is an appropriate general method.

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