Growth of Algebras

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Some Definitions and Notation

Throughout, $F = \mathbb{R}$ or $\mathbb{C}$ and $0 \in \mathbb{N}$.

Definition

Let A be a vector space over F equipped with an additional binary operation from $A \times A$ to A, denoted here by $\cdot$ (i.e. if x and y are any two elements of A, $x \cdot y$ is the product of x and y). Then A is an algebra over F (a F-algebra) if the following hold for all elements x, y, and z in A, and all elements a and b in F:

- $(x + y) \cdot z = x \cdot z + y \cdot z$
- $x \cdot (y + z) = x \cdot y + x \cdot z$
- $(ax) \cdot (by) = (ab) \cdot (xy)$.

More Definitions and Notation

Definition

Let A be an F-algebra. We say that A is finitely generated provided there is $\{a_1, a_2, \cdots, a_r\} \subseteq A$ such that every element of A can be written as a finite linear combination of monomials in $a_1, a_2, \ldots, a_r$. V will denote the F-span of $\{a_1, a_2, \ldots, a_r\}$. V is called a finite dimensional generating subspace (fdgs) for A.

Subspaces of Interest

Definition

Let A be an F-algebra with finite dimensional generating subspace $V = \text{span}\{a_1, a_2, \ldots, a_r\}$. The length of a monomial in A is the number of letters that make up the monomial, counting repetitions. Define $V^0 = F$ and for $n \geq 1$, V^n as the F-span of monomials in $a_1, \ldots, a_r$ of length n and $A_n = \sum_{i=0}^{n} V^i$.

Proposition

For the A_n's as defined above, $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ is an ascending chain of finite dimensional subspaces of A and $A = \bigcup_{n=0}^{\infty} A_n$.

Definition of a Growth Function for an Algebra

Definition

Define a growth function of A with respect to V, $d_V : \mathbb{N} \to \mathbb{N}$ by $d_V(n) = \dim(A_n) = \dim(\sum_{i=0}^{n} V^i)$.

Question

What types of functions can these growth functions be?

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- The growth function $d_V(n) = \dim(A_n) = \dim(\sum_{i=0}^{n} V^i)$.
- Each $V^n = span\{x^n\}$, so $\{x^n\}$ is a basis for V^n.
- Since $\{1, x, \ldots, x^n\}$ is a basis for polynomials of at most degree n, $d_V(n) = \dim(A_n) = n + 1$.

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- $d_V(n) = \sum_{i=0}^{n}(i + 1) = \frac{n^2+3n+2}{2}$.

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- fdgs: $V = span\{x, y\}$.
- Each V^n has 2^n basis elements since there are two choices for each letter of a monomial of length n.
- Thus $d_V(n) = \sum_{i=0}^{n} 2^i = 2^{n+1} - 1$.

Ideals, Free Algebras, Representation

Definition

A subspace I of A is called an ideal if for all $a \in A$ and $x \in I$, $ax \in I$ and $xa \in I$.

Theorem

Every finitely generated algebra is isomorphic to a quotient of a finitely generated free algebra. In particular, $A \approx F\langle x_1, x_2, \ldots, x_r\rangle / I$, for some ideal I of $F\langle x_1, x_2, \ldots, x_r\rangle$.

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- We can view elements of I as "zero".
- In order to calculate the growth function for various finitely generated algebras, we may calculate them for quotients of finitely generated free algebras.

Ideals Generated by Monomials

- In particular, we will look at quotients whose ideals are generated by finitely many monomials in $x_1, x_2, \ldots, x_r$. We will refer to monomials as words and denote them by $m_1, m_2, \ldots, m_k$.

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- An ideal generated by the set $\{m_1, m_2, \dots, m_k\}$ is the set of linear combinations of monomials who contain at least one of $m_1, m_2, \dots, m_k$ as a factor (subword) denoted $I = (m_1, m_2, \dots, m_k)$. Such ideals are called <u>monomial ideals</u>.

- From now on, we will let $A = F\langle x_1, x_2, \ldots, x_r\rangle / I$ where I is a monomial ideal.
- Since words in I are considered zero, every element of A can be written as a linear combination of words not in I.
- Let $\mathcal{B}$ be the collection of words not in I including 1, i.e., $\mathcal{B}$ consists of the words that do not have any of $m_1, m_2, \ldots, m_k$ as a subword.

Proposition

$\mathcal{B}$ is a basis for A.

- $V = \text{span}\{x_1, x_2, \ldots, x_r\}$ is a fdgs.
- $V^n =$ the span of words in $\mathcal{B}$ of length n.
- So, $\dim V^n =$ number of words in $\mathcal{B}$ of length n.
- Since $A_n = \sum_{i=0}^{n} V^i$ and $\mathcal{B}$ is a basis for A, $\dim A_n =$ the number of words in $\mathcal{B}$ of length at most n.

Example

Determine a growth function for $\mathbb{R}\langle x, y\rangle / I$ where $I = (xy)$.

- Any word with xy as a subword is zero.

n	**Words in $\mathcal{B}$ of length n**
0	1
1	x, y
2	x^2, y^2, yx
3	x^3, y^3, y^2x, yx^2

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- Given $n \geq 1$, there is only one word of length n in $\mathcal{B}$ beginning with x, namely x^n. There are n such words beginning with y, namely $y^k x^{n-k}$ for $1 \leq k \leq n$.

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- Given $n \geq 1$, there is only one word of length n in $\mathcal{B}$ beginning with x, namely x^n. There are n such words beginning with y, namely $y^k x^{n-k}$ for $1 \leq k \leq n$.
- So there are $n+1$ words of length n in $\mathcal{B}$, i.e., $\dim V^n = n+1$. Thus, $d_V(n) = \sum_{i=0}^{n} (i+1) = \frac{n^2+3n+2}{2}$.

We need a better way to count our words. One way involves using a directed graph.

Definition

A directed graph is a set V of vertices with a set E of ordered pairs of vertices called arrows.

Definition

Let u, v be words. We say u is a prefix of v provided there is a word w for which $v = uw$. We say u is a suffix of v provided that there is a word z for which $v = zu$.

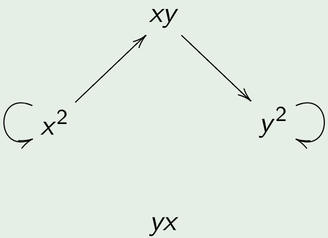
Example

x^2y is a prefix of x^2y^3x and yx is a suffix of x^2y^3x

- Let $d+1$, where $d \geq 2$, be the maximum length of the generators in I and $\{w_1, w_2, \ldots, w_k\}$ be words in $\mathcal{B}$ of length d. We use this set of words as vertices for a directed graph.
- We draw an arrow from w_i to w_j provided there is a word in $\mathcal{B}$ of length $d+1$ whose prefix of length d is w_i and whose suffix of length d is w_j. We will call our graph the overlap graph for $\mathcal{B}$, and denote it by Γ.

Example

$I = (yx^2, y^2x, xyx, yxy)$
$d + 1 =$ maximum length of generators in $I = 3$
$d =$ max length$-1 = 2$.
vertices: x^2, y^2, xy, yx
$x^2 \to xy$ provided there is a word of length 3 in $\mathcal{B}$ whose prefix is x^2 and suffix is xy.
Words of length 3 in $\mathcal{B}$: x^3, y^3, x^2y, xy^2



Cycles

Definition

A path in a directed graph is a sequence of arrows in the same direction. We call path $u_1 \to u_2 \to \cdots \to u_t \to u_1$ a cycle provided $u_i \neq u_j$ for $i \neq j$. The length of a path is the number of arrows in it.

Proposition

Each path of length j, for $j \geq 0$, corresponds to a unique word in $\mathcal{B}$ of length $d + j$. Each word in $\mathcal{B}$ of length $d + j$ corresponds to a unique path in our graph with j arrows.

Example

path	**word**
$x^2 \longrightarrow xy$	x^2y
$x^2 \longrightarrow xy \longrightarrow y^2$	x^2y^2

Theorem (Ufnarovski)

If Γ has two intersecting cycles, then the growth function for A is exponential.
If Γ has no intersecting cycles, then the growth function for A is bounded above and below by two polynomials of degree s where s is the maximal number of distinct cycles on a path in Γ.

Example Revisited

Example

$I = (yx^2, y^2x, xyx, yxy)$
$d + 1 =$ maximum length of generators in $I = 3$
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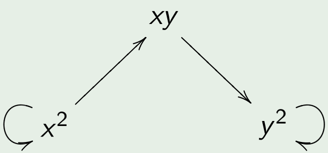
vertices: x^2, y^2, xy, yx
The overlap graph for $\mathcal{B}$ has two cycles, so the growth function is bounded by a polynomial of degree 2.



Exponential Growth

It is known that growth functions for our algebras are either exponential or polynomial. We would like to know more specifically, for a given d, what types of growth functions are attainable.

Proposition

For some ideal I generated by words of at most length $d+1$, the corresponding algebra $F\langle x, y\rangle / I$ has exponential growth.

Proof.

Consider $I = (y^{d+1})$. Then the following cycles intersect: $x^d \rightarrow x^d$ and $x^d \rightarrow x^{d-1}y \rightarrow x^{d-2}yx \rightarrow x^{d-3}yx^2 \rightarrow \cdots \rightarrow yx^{d-1} \rightarrow x^d$. So by Ufnarovski's Theorem, $F\langle x, y\rangle / I$ has exponential growth. $\square$

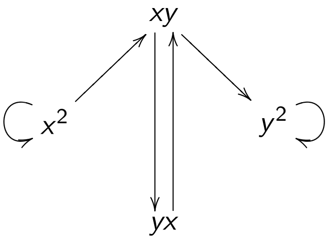
Dr. Ellingsen's Conjecture

Conjecture (Dr. Ellingsen's)

If I is generated by words of at most length $d+1$, then the growth function is either exponential or is bounded by a polynomial with degree at most $d+1$.

$d = 2$

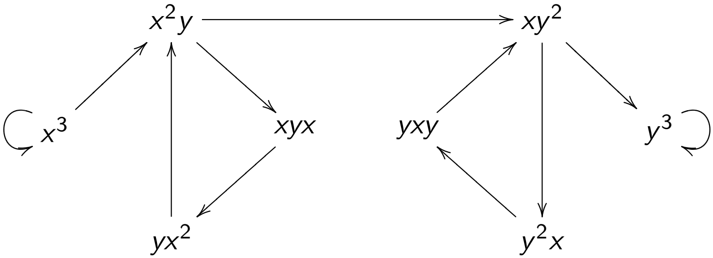
We have shown for $d = 2$ that the growth function must be either exponential or bounded by a polynomial of degree at most 3.



$I = (y^2x, yx^2)$

$d = 3$

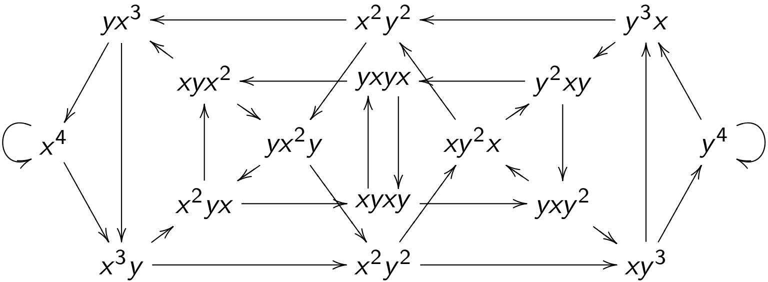
Additionally, we have shown that for $d = 3$, the growth function must be either exponential or bounded by a polynomial of degree at most 4.



$I = (yx^4, xyxy, yxyx, y^2x^2, y^3x)$

$d = 4$

What about $d = 4$?



$d = 4$

	yx^3			x^2y^2			y^3x	
		xyx^2		$yxyx$		y^2xy		
x^4			yx^2y		xy^2x			y^4
		x^2yx		$xyxy$		yxy^2		
	x^3y			x^2y^2			xy^3	

$d = 4$

yx^3 $\quad$ x^2y^2 $\quad$ y^3x

xyx^2 $\quad$ $yxyx$ $\quad$ y^2xy

x^4 $\quad$ yx^2y $\quad$ xy^2x $\quad$ y^4

x^2yx $\quad$ $xyxy$ $\quad$ yxy^2

x^3y $\quad$ x^2y^2 $\quad$ xy^3

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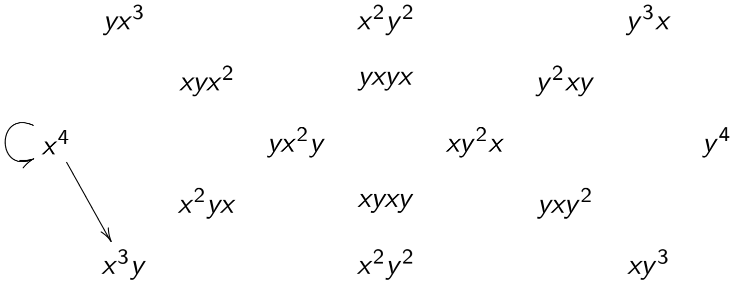
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xy^3

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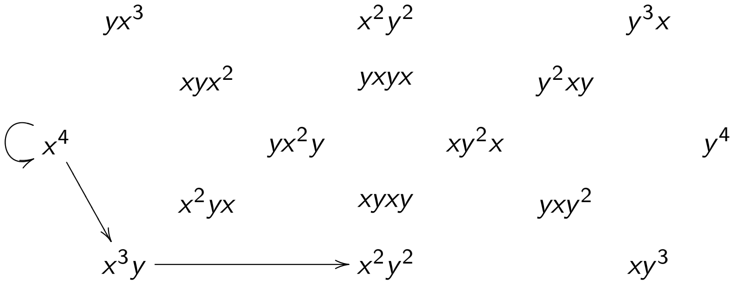
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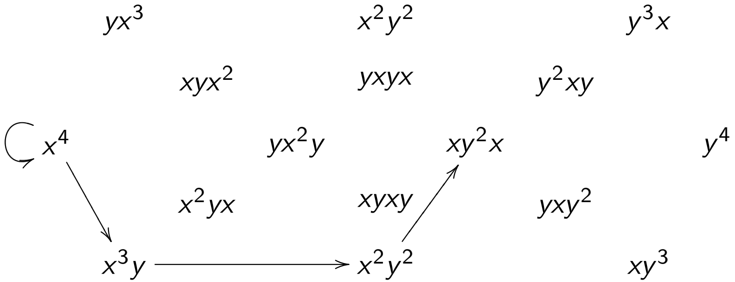
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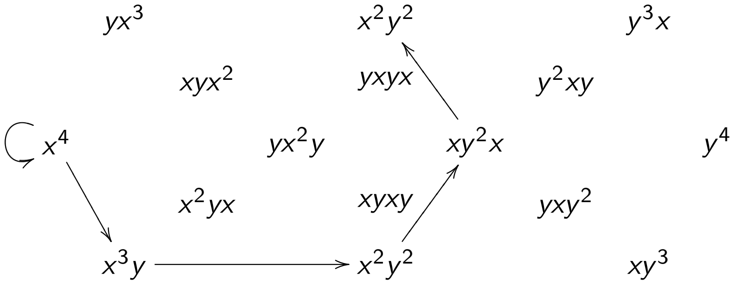
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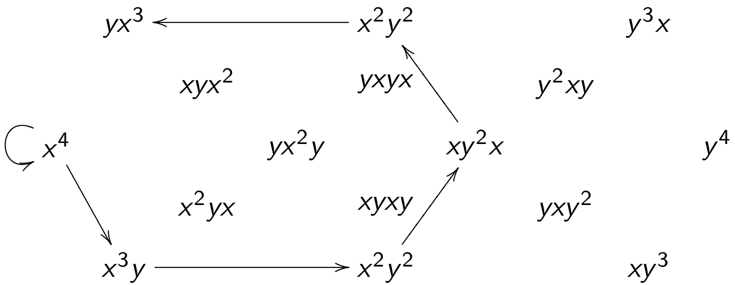
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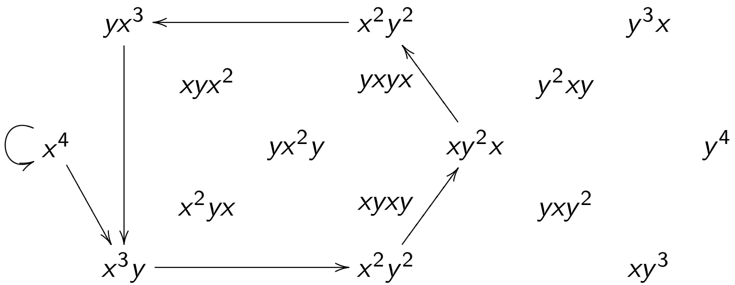
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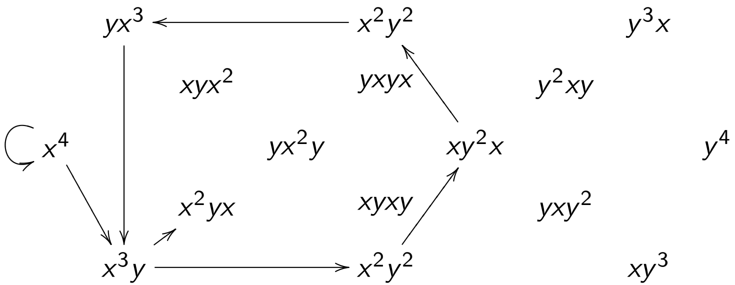
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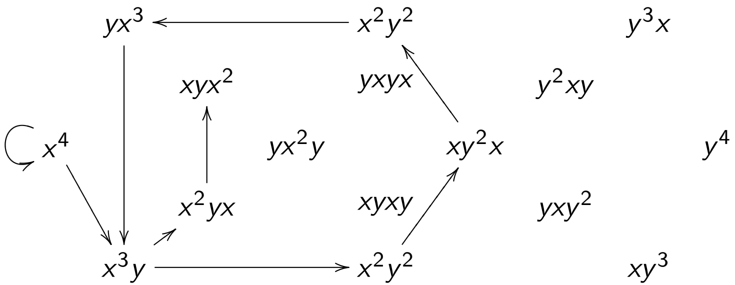
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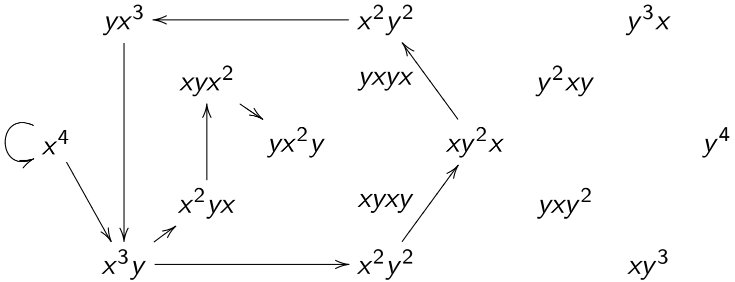
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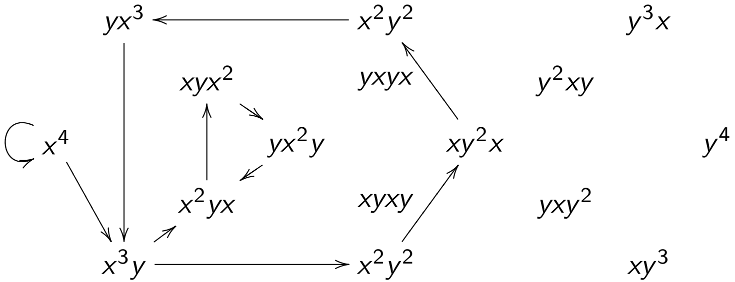
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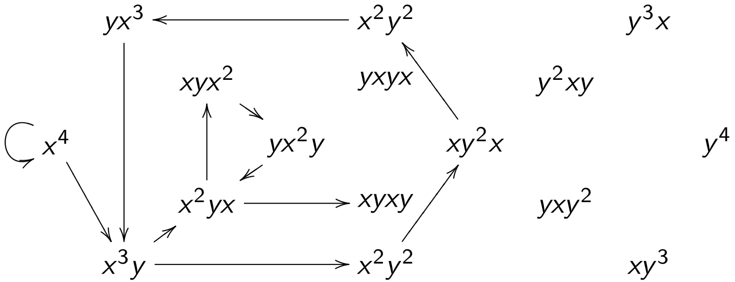
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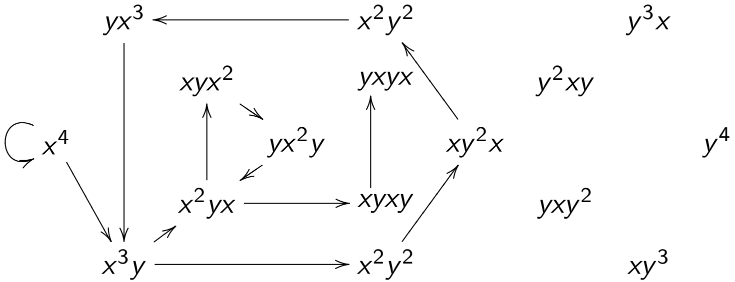
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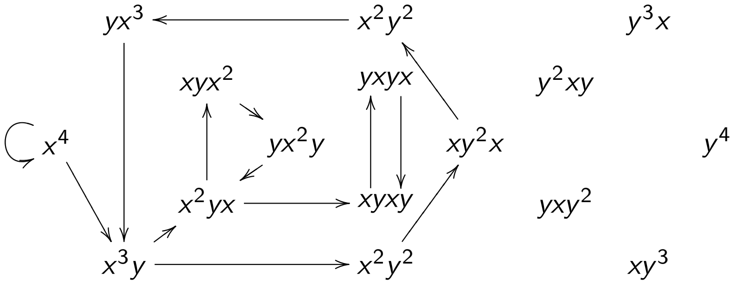
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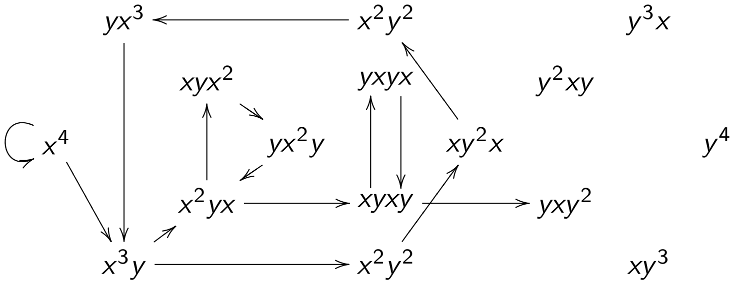
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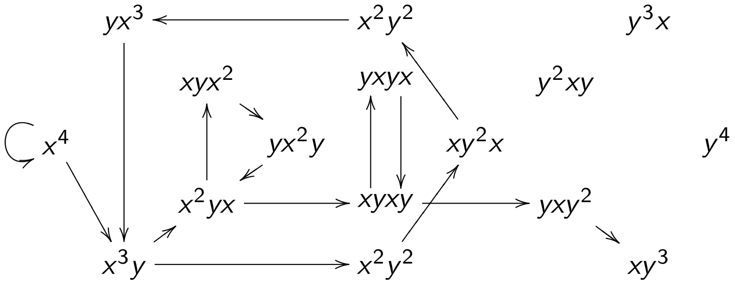
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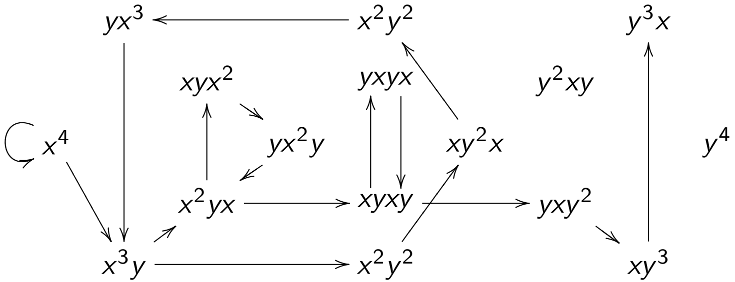
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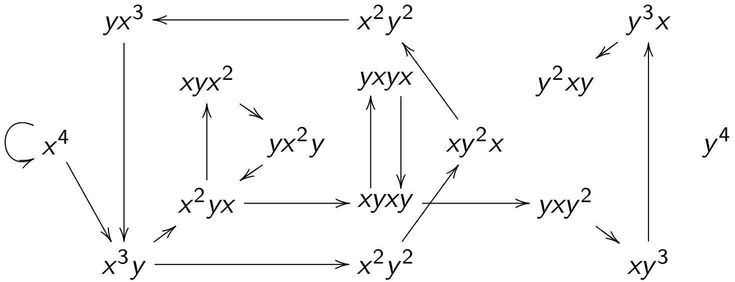
x^4
yx^2y
xy^2x
y^4
x^2yx
$xyxy$
yxy^2
x^3y
x^2y^2
xy^3

$d = 4$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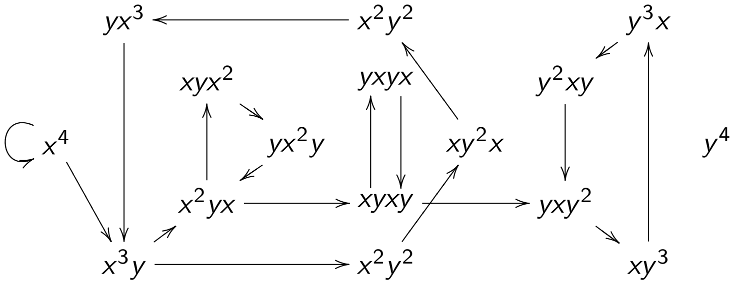
x3y
x2y2
xy3

$d = 4$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$yxyx$
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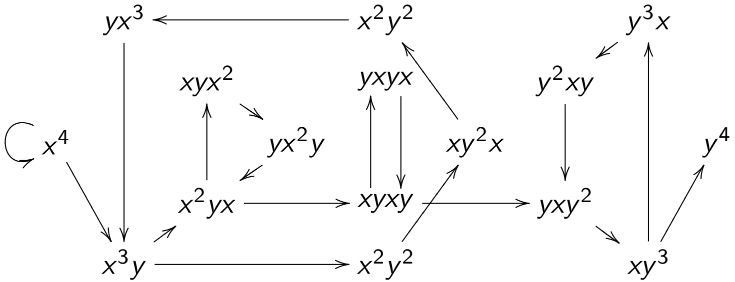
xy^3

$d = 4$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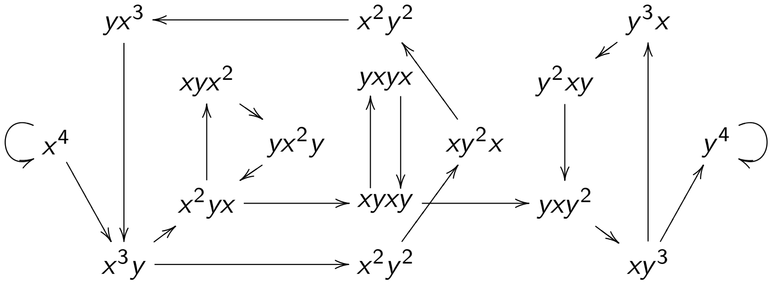
$d = 4$


yx^3
x^2y^2
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$yxyx$
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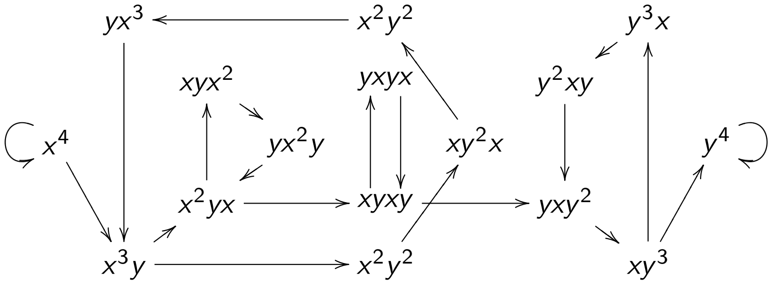
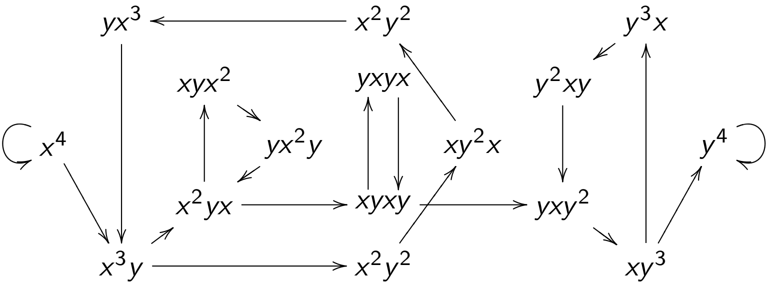
yxy^2
x^3y
x^2y^2
xy^3

$d = 4$



$$I = (yx^4, xyx^3, yxyx^2, y^2x^2y, yx^2y^2, x^2y^3, yxy^2x, y^2xyx, y^3x^2, xy^2xy, y^4x)$$

$d = 4$



$I = (yx^4, xyx^3, yxyx^2, y^2x^2y, yx^2y^2, x^2y^3, yxy^2x, y^2xyx, y^3x^2, xy^2xy, y^4x)$

Thus, the conjecture fails for $d = 4$ because of the 6 cycles!

High Upper Bound

We would like to look at maximum possible degrees of polynomial growth functions.

Theorem (Ellingsen)

If there are $d + i$ words of length d, the growth function is either exponential or bounded by a polynomial of degree $i + 1$.

This gives us a really high upper bound on the possible degrees for our growth functions. There are 2^d words of length d, which we can write as $d + (2^d - d)$ words, so the growth of our algebra with corresponding ideal generated by words of length at most $d + 1$ is either exponential or bounded by a polynomial of degree $2^d - d + 1$.

Definitions

Definition

Let v be a word of length p and w a word of length $d \geq p$. w is periodic provided w is a prefix of v^j from some positive integer j. We call v a base for w and the length p is a period for w. The smallest possible period is the minimal period.

Example

1.) Let $w = x^2yx^2yx$. Then w has minimal period 3 with base x^2y. Note that w also has period 6 with base x^2yx^2y.
2.) Let $u = x^2yx^2$. Interestingly u has periods 3 and 4 with bases x^2y and x^2yx respectively.

Definitions

Definition

Let $w = a_0 a_1 \dots a_{d-1}$ be a word of length d. Then any word of the form $a_i a_{i+1} \dots a_{d-1} a_0 \dots a_{i-1}$ is called a cyclic permutation of w.

Note that we can draw an arrow from any word to exactly one cyclic permutation of itself, namely
$a_0 a_1 \dots a_{d-1} \longrightarrow a_1 a_2 \dots a_{d-1} a_0$.

Example

Let $w = xy^2xy$. Then the cyclic permuations of w are $xy^2xy, y^2xyx, yxyxy, xyxy^2, yxy^2x$. Note these all connect and give us a cycle: $xy^2xy \rightarrow y^2xyx \rightarrow yxyxy \rightarrow xyxy^2 \rightarrow yxy^2x \rightarrow xy^2xy$.

Lemma

Let w be a word of length d. If the minimal period of w is d, then w and its cyclic permutations form a cycle of length d.

Proposition

For some ideal I generated by words of length at most $d+1$, the corresponding algebra has growth function of degree $d+1$.

Proof.

Consider the path $x^d \to x^{d-1}y \to x^{d-2}y^2 \to \cdots \to x^2y^{d-2} \to xy^{d-1} \to y^d$. We have cycles of length 1 at x^d and y^d. Let $1 \leq i \leq d-1$. Each $x^{d-i}y^i$ has period d. By the lemma, they are on cycles of length d. Each vertex on a cycle has $d-i$ x's and the different number of x's makes the cycles distinct. □

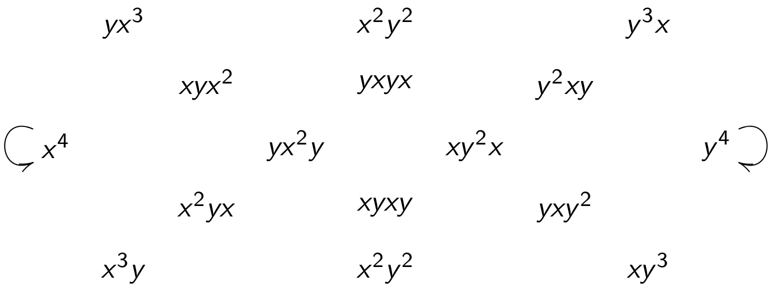
Case $d = 4$

We would like to know the maximum possible degree that is attainable for $d = 4$. We can do this by putting as many distinct cycles on a path as possible by using the smallest cycles first. For $d = 4$, there are $2^4 = 16$ possible vertices to use in cycles. We want to start by finding all the cycles which contain only one vertex, namely, x^4 and y^4. By exhaustion, we can find all cycles containing 2, 3, and 4 vertices.

Number of vertices in a cycle	Number of cycles
1	2
2	1
3	2
4	3

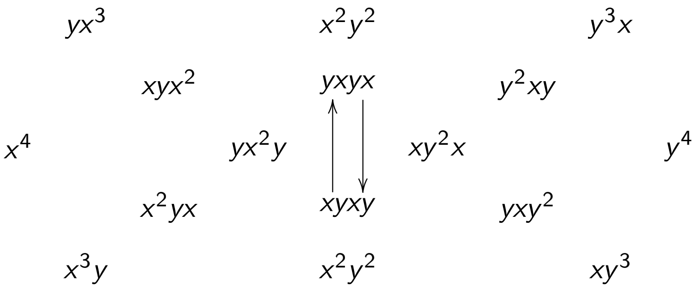
Case $d = 4$

- Two distinct cycles with one vertex



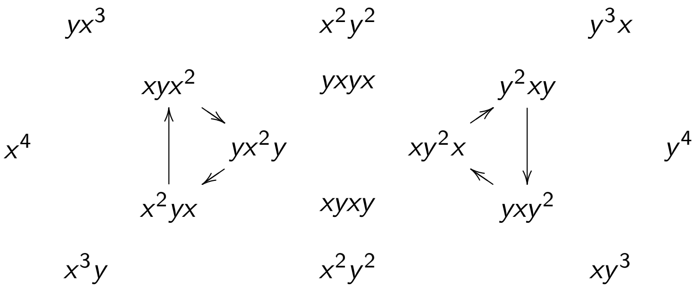
Case $d = 4$

- One distinct cycle with two vertices



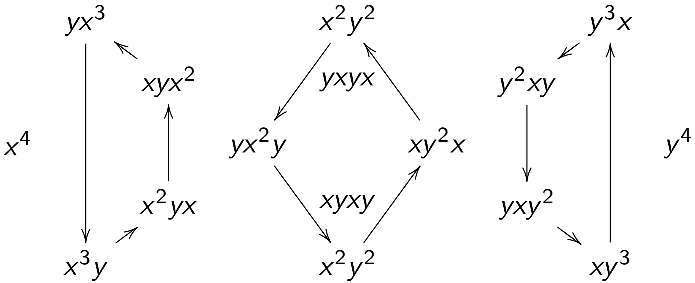
Case $d = 4$

- Two distinct cycles with three vertices



Case $d = 4$

- Three distinct cycles with four vertices



Case $d = 4$

By using two cycles with 1 vertex, one cycle with 2 vertices, two cycles with 3 vertices, and one cycle with 4 vertices, we use 14 out of the total 16 possible vertices $1(2) + 2(1) + 3(2) + 4(1) = 14$. Thus, we could potentially connect these 6 cycles in a path which would correspond to a maximum possible degree of 6 for the growth function.

Counting Cycles

We need a better way to count cycles of small lengths.

Lemma

Let w be a word of length d. If w has a minimal period $p \leq d$, w is a vertex on a cycle of length p. Additionally, every vertex on a cycle of length $p \leq d$ must be periodic with period of length p. Moreover, the bases of length p for any two words on these cycles are cyclic permutations of each other.

Case $d = 5$

Using the previous lemma, we are able to count the cycles with up to 5 vertices.

Number of vertices in a cycle	Number of cycles
1	2
2	1
3	2
4	3
5	≥ 4

Similarly to the $d = 4$ case, we can count the number of distinct cycles that we can put in a path using only $2^5 = 32$ vertices. $1(2) + 2(1) + 3(2) + 4(3) + 5(2) = 32$. This gives us an upper bound of 10 cycles.

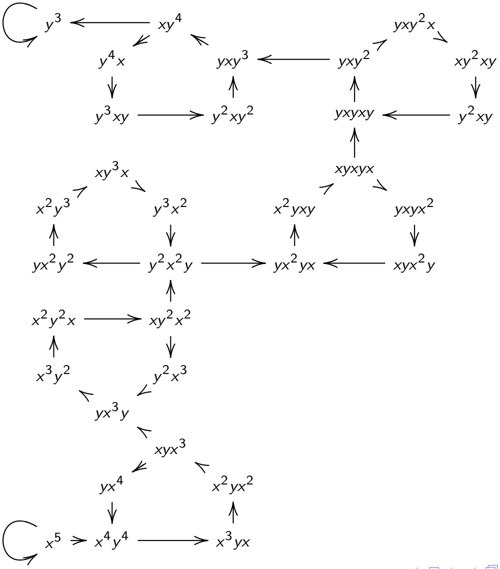
Prime Cyclic Permutation

Proposition

For d prime, there are $\frac{2^d - 2}{d}$ disjoint cycles of length d.

Example

- For $d = 5$, we have $\frac{2^5 - 2}{5} = 6$ cycles of length 5.
- We have connected all 6 cycles of length 5 on a path.


y^3
xy^4
y^4x
yxy^3
yxy^2x
yxy^2
xy^2xy
y^3xy
y^2xy^2
$yxyxy$
y^2xy
$xyxyx$
xy^3x
x^2y^3
y^3x^2
x^2yxy
$yxyx^2$
yx^2y^2
y^2x^2y
yx^2yx
xyx^2y
x^2y^2x
xy^2x^2
x^3y^2
y^2x^3
yx^3y
xyx^3
yx^4
x^2yx^2
x^5
x^4y^4
x^3yx

- We have also done this for $d = 7$ and obtained a growth of degree 20!
- We are currently working on finding an algorithm that allows us to do this for any d prime.
- We are also looking for a better way to count the cycles of small lengths and use them to find upper bounds on the degrees of our growth functions.

Conjecture

For d prime, all of the $\frac{2^d - 2}{d}$ cycles of length d can be connected on a path.

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- Krause, G.R. and T.H. Lenagan. *Growth of algebras and Gelfand-Kirillov dismension*, volume 116 of *Research Notes in Mathematics*. Pitman Publishing Inc., London, 1985.
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