MATRIX COMPLETION PROBLEM

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Abstract. A matrix completion problem involves completing a partially specified matrix to satisfy a given property. The focus of this paper is completing the partially specified matrix so that it will commute with a fully specified matrix. In particular, given a fully specified matrix A, and a partially specified matrix X, when can we complete the remaining entries in X so that the equation $AX - XA = 0$ will be satisfied? The three approaches used to complete matrices are the Polynomial Approach, the Matrix Equation Approach, and the Graph Theoretic Approach. The main theorem classifies all admissible patterns for a Jordan block. This allows us to identify all patterns in a partially specified matrix X such that X can be completed to commute with a Jordan block. The Classification Theorem is also extended to matrices with multiple Jordan blocks and matrices that are permutation similar to a Jordan block.

1. Introduction

Matrix completion problems explore whether partially specified matrices can be completed in a strategic way so that the completed matrix has a given property. Some examples of linear matrix equations involving completions are: AX=B, AXB=C, AX+YB=0, AX+XB=0 and AX-XA=0 where X and Y are partially specified matrices, and A and B are the given fully specified matrices. The last equation is of special interest since it is another way of writing AX=XA which defines the commutative property between matrices. If X has no specified entries, then we can choose X to be the identity matrix, A-1 if it exists, a power of A or a polynomial of A, and it will commute with A. This is the easiest case. However, what if we one entry is specified? What about 2? What if n entries in X are specified?

We explored the question: under what circumstances can the unspecified entries of a partial matrix X be chosen so that X commutes with a fully specified matrix A? Or, when can X be completed so that AX-XA=0? Geoffrey Buhl (1996) attempted to answer this question using a Polynomial Approach and a Matrix Equation Approach. Buhls results and approaches were extensively used during our research. We used the Polynomial Approach, the Matrix Equation Approach as well as a less developed Graph Theory Approach. The Graph Theory Approach involved using graph theory methods used by Leslie Hogben (2000).

Using these three methods, our research group created and proved numerous results involving admissible patterns for Jordan blocks, permutations of Jordan blocks, matrices composed of direct sums and matrices in Jordan canonical form. The Classification Theorem gives all the admissible patterns of specified entries for a partial matrix X that allow it to be completed to commute with a

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matrix A when A is a Jordan block. Using the results from the Classification Theorem, we derived new admissible patterns for partial matrices that commute with a matrix A when A is permutation similar to a Jordan block. We built admissible patterns for a matrix equal to the direct sum of two matrices. There is a peek at the general solution for the set of matrices that commute with a given matrix A. The conditions under which X will commute with A are explored throughout this paper.

2. Prelimaries

Definition 2.1. A *partial matrix pattern* for an $m \times n$ matrix is a set of specified entry locations $\alpha = \{(i_1, j_1), \ldots, (i_k, j_k)\} \subseteq \{(i,j) | 1 \leq i \leq m \text{ and } 1 \leq j \leq n\}$.

Definition 2.2. Given a partial matrix pattern α, an *α-partial matrix* A is a matrix where a_{ij} is a specified entry if and only if $(i,j) \in \alpha$.

In general the term 'admissible' is used to describe a partial matrix pattern that has a completion that satisfies the given property of interest. Our property of interest is commutativity with a given matrix A.

Definition 2.3. A partial matrix pattern α is an *admissible pattern* for an $n \times n$ matrix A if any α-partial matrix X has a completion $\hat{X}$ that commutes with A.

Definition 2.4. A partial matrix pattern α is a *maximal* admissible pattern for an $n \times n$ matrix A if there exists no admissible patterns with size strictly larger than $|\alpha|$.

Definition 2.5. Let $A \in M_{m,n}(\mathbb{F})$, $\alpha \subseteq \{1, \ldots, m\}$ and $\beta \subseteq \{1, \ldots, n\}$. The matrix A_β is the $m \times |\beta|$ submatrix of A lying in the columns β. The matrix ${}_\alpha A$ is the $|\alpha| \times n$ submatrix of A lying in the rows α.

Definition 2.6. Let $A \in M_{n,n}(\mathbb{F})$, $\alpha \subseteq \{1, \ldots, n\}$ and $\beta \subseteq \{1, \ldots, n\}$. The matrix $A[\alpha, \beta]$ is the $|\alpha| \times |\beta|$ submatrix of A obtained by deleting entries (i,j) if i is not in α or j is not in β. If $\alpha = \beta$, denote this principle submatrix $A[\alpha]$.

Definition 2.7. Let $A = (a_{ij}) \in M_{m,n}(\mathbb{F})$, then the vector $\text{vec}(A) \in (F)^{mn}$ is defined as $\text{vec}(A) = [a_{11}, \ldots, a_{m1}, a_{12}, \ldots, a_{m2}, \ldots, a_{1n}, \ldots, a_{mn}]^T$.

This is an invertible linear transformation from $M_{m,n}(\mathbb{F})$ to $(F)^{mn}$ given by the standard basis element of $M_{m,n}$,E_{ij}, being mapped to the standard basis vector for $(F)^{mn}$, $e_{i+(j-1)m}$. This conversion from matrix positions to the vec ordering is used often, using the formula $\text{vec}(i,j) = i + (j-1)m$ for the (i,j) position in a $m \times n$ matrix.

Definition 2.8. The dimension of the eigenspace of $A \in M_n(\mathbb{F})$ corresponding λ is the *geometric multiplicity* and is denoted $m_g(\lambda)$. The multiplicity of λ as a zero of the characteristic polynomial is the *algebraic multiplicity* and is denote $m_a(\lambda)$.

Definition 2.9. A $n \times n$ matrix A is *nonderogatory* if the geometric multiplicity of each distinct eigenvalue is one.

Definition 2.10. S_i is the set of columns from $\Omega(A)$ corresponding to the entries of the diagonal. In general, we have:

$$S_i = \{c_{1+(n-1-i)n+j(n+1)} | 1 \leq j \leq i\}$$

.

An important characterization of nonderogatory matrices is the following: A matrix A commutes with a nonderogatory matrix B if and only if B can be written as a polynomial in A [**?**].

There are three approaches to the commutative matrix completion problem: polynomial, matrix equation, and graph theoretic.

2.1. **Polynomial Completions.** The polynomial approach assumes that the given matrix A is nonderogatory, and attempts to

Theorem 2.11. *B's theorem for admissible patterns*

Theorem 2.12. *B's theorem for maximal admissible patterns*

2.2. **Matrix Equations Completions.**

Theorem 2.13. *B's theorem for admissible patterns*

Theorem 2.14. *B's theorem for maximal admissible patterns*

2.3. **Graph Theoretic Completions.** The graph theoretic approach first appeared in [4] and uses marked directed graphs (mardigraphs) to classify admissible patterns.

3. Patterns for one Jordan Block

Using the three approaches to the commutative completions, we characterize admissible patterns for a Jordan block.

3.1. **Polynomials.**

Definition 3.1. Let V_d be an $n \times n$ matrix such that the (i, j) is 1 if $j - i + 1 = d$ and 0 otherwise for $1 \leq d \leq n$.

Definition 3.2. The d diagonal refers to the (i, j) position where $d = j - i + 1$

Definition 3.3. Let $\mathbf{v_d} = \text{vec}(V_d)$

Lemma 3.4. *Vectors* $\mathbf{v_1}...\mathbf{v_n}$ *are linearly independent.*

Proof. Recall that $\mathbf{v_d} = \text{vec}(V_d)$ and V_d has 1's in the d^{th} diagonal. Since diagonals do not share entry positions, there will be no non-zeros entries in the same positions. Thus $\mathbf{v_1}...\mathbf{v_n}$ are linearly independent vectors since none can be written as a linear combination of the other vectors. □

Lemma 3.5. $V_2^p = V_{p+1}$ *for* $0 \leq p \leq n - 1$

Proof. For the base case when $p = 0$

$$V_2^0 = I = V_1 = V_{0+1} \tag{3.1}$$

Thus the lemma holds for the base case since all non-zero entires in I are in positions where $i = j$ or $d = j - i + 1 = 1$ and so $V_1 = I$

Now assume $V_2^{p-1} = V_p$ for $1 \leq k \leq n$ Then $V_2^p = V_2 V_2^{p-1} = V_2 V_p$ Let e_p be a standard basis vector, then V_2 is in the form:

$$\begin{bmatrix} - & \mathbf{e_2^T} & - \\ - & \mathbf{e_3^T} & - \\ & \vdots & \\ - & \mathbf{e_n^T} & - \\ - & \mathbf{0^T} & - \end{bmatrix} \tag{3.2}$$

while V_p is in the form:

$$\begin{bmatrix} | & & | & | & & | \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{e_1} & \dots & \mathbf{e_{n-p+1}} \\ | & & | & | & & | \end{bmatrix} \tag{3.3}$$

where there are $p-1$ columns of 0 before the e_1 column. Thus

$$V_2V_p = \begin{bmatrix} - & \mathbf{e_2^T} & - \\ - & \mathbf{e_3^T} & - \\ & \vdots & \\ - & \mathbf{e_n^T} & - \\ - & \mathbf{0^T} & - \end{bmatrix} \begin{bmatrix} | & & | & | & & | \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{e_1} & \dots & \mathbf{e_{n-p+1}} \\ | & & | & | & & | \end{bmatrix} \tag{3.4}$$

When multiplying, the first matrix's rows are multiplied by the second matrix's columns. so letting r_i be the i^{th} row of V_2 and letting c_j be the j^{th} column of V_k we know that entry $(i,j) = r_ic_j$. Both matrices can be expressed in terms of the standard basis vectors so $e_s^Te_t$ is equal to 1 if $s=t$ and 0 if $s \neq t$. Since $r_1 = e_2^T$ and $c_d = e_2$, the $(1,d)$ entry is a 1. In general $r_q = e_{q+1}$ and $c_{d+q} = e_{q+1}$ where $1 \leq q \leq n-1$. Then $e_s^Te_t = 1$ for all (i,j) entries where $i=q$ and $j = d+q$. Thus $j-i+1 = (d+q)-(q)+1 = d+1$. Therefore since all 1's will be in the $d+1$ diagonal, $v_2^d = V_2V_2^{d-1} = V_{d+1}$ □

Lemma 3.6. *Let J be an $n \times n$ Jordan block. Then the $k+1$ column of $\Psi(J)$ can be expressed as linear combinations of vectors $\mathbf{v_1}, ..., \mathbf{v_n}$:*

$$\mathbf{c_{k+1}} = \sum_{s=0}^{k} \binom{k}{s} \lambda^{k-s} \mathbf{v_{s+1}} \tag{3.5}$$

where $0 \leq k \leq n-1$

Proof. Since J is a Jordan block, it can be written as a linear combination of V_1 and V_2:

$$J = \lambda V_1 + V_2$$

V_1 commutes with V_2^b for any b since $V_1 = I$ and thus we can use binomial theorem to express the powers of J as linear combinations of the powers V_1 and V_2. Thus

$$J^k = (\lambda V_1 + V_2)^k \tag{3.6}$$

$$= \sum_{s=0}^{k} \binom{k}{s} (\lambda V_1)^{k-s} V_2^s \tag{3.7}$$

$$= \sum_{s=0}^{k} \binom{k}{s} (\lambda)^{k-s} V_2^s \tag{3.8}$$

By Lemma 3.5, $V_2^s = V_{s+1}$ so

$$J_k = \sum_{s=0}^{k} \binom{k}{s} (\lambda)^{k-s} V_{s+1} \tag{3.9}$$

Since vec is a linear operator and $\mathbf{v_k} = \text{vec}(V_k)$,

$$\mathbf{c_{k+1}} = \text{vec}(J^k) = \sum_{s=0}^{k} \binom{k}{s} (\lambda)^{k-s} \text{vec}(V_{s+1}) = \sum_{s=0}^{k} \binom{k}{s} (\lambda)^{k-s} \mathbf{v_{s+1}} \tag{3.10}$$

where $0 \leq k \leq n-1$.

□

Theorem 3.7. *Vectors $\mathbf{v_1}...\mathbf{v_p}$ form a basis for the columnspace of $\Psi(J)$*

Proof. Let the dimension of the columnspace of $\Psi(J)$ be p where $1 \leq p \leq n$. Then p linearly independent vectors are needed to form the basis for columnspace of $\Psi(J)$. By lemma 3.6 we will have n vectors in the linear combination making up the n column of $\Psi(J)$. Since $\mathbf{v_1}...\mathbf{v_p} \subseteq \mathbf{v_1}...\mathbf{v_n}$ we know by Lemma 3.4 that these vectors are linearly independent. Therefore since we have p linearly independent vectors that are in the span of the columnspace, vectors $\mathbf{v_1}...\mathbf{v_p}$ form a basis for the columnspace of $\Psi(J)$. □

Theorem 3.8. *Let J be a Jordan block, then the following statements are equivalent:*

(a) The columns of $\Psi(J)$ for $0 \leq k \leq n-1$ can be expressed as:

$$\mathbf{c_{k+1}} = \sum_{s=0}^{k} \binom{k}{s} \lambda^{k-s} \mathbf{v_{s+1}} \tag{3.11}$$

(b) The rows of $\Psi(J)^T$ for $1 \leq k \leq n$ can be expressed as:

$$\mathbf{r_k} = \sum_{s=1}^{k} \binom{k-1}{s-1} \lambda^{k-s} \mathbf{v_s} \tag{3.12}$$

(c) The i^{th} entry in each row for the k^{th} equivalence class for $1 \leq i \leq n$ can be expressed as:

$$\binom{i-1}{k-1} \lambda^{i-k} \tag{3.13}$$

(d) The (i,j) entry of J^k can be written as

$$\binom{k}{j-i} \lambda^{k-(j-i)} \tag{3.14}$$

if $j-i < 0$ then then entry is zero

Proof. This shows that (a) implies (b) Since the columns of $\Psi(J)$ become the rows of $\Psi(J)^T$ it follows that the formula for the rows of $\Psi(J)^T$ is the same as the formula for the columns of $\Psi(J)$. Starting the index at $s = 1$ instead of $s = 0$ will merely shift all indexed items as well as the range of the index down by one. Using the same logic we can show the equivalence in the reverse direction.

This shows that (b) implies (c) Since the $\mathbf{v_s}$ term is derived from V_s term where there is a 1 in the entry when $j - i + 1 = s$ and a 0 in all other entries. Each equivalence class occurs when $k = i + (j-1)n$ which is exactly when $\mathbf{v_s}$ has a non-zero entry, more specifically, when the entry is one. Thus, each entry of the row is going to be the same as the coefficient of the corresponding vector in the rows of $\Psi(J)^T$. The coefficient for $\mathbf{v_s}$ is

$$\binom{k-1}{s-1} \lambda^{k-s} \tag{3.15}$$

The variables, however, must be changed. The k which referred to which column of $\Psi(J)$ or row of $\Psi(J^T)$ was being looked at changes to an i since i refers to the entry in the row of $\Psi(J)$ or essentially, the column of $\Psi(J)$. The s changes to a k since s was the index and referred to which entry in the column of $\Psi(J)$ and now becomes k to refer to which equivalence class or row of $\Psi(J)$. The formula then becomes:

$$\binom{i-1}{k-1}\lambda^{i-k} \tag{3.16}$$

Using the same logic we can show equivalence in the reverse direction.

(a) implies (d) Since $\mathbf{c_{k+1}} = \text{vec}(J^k)$, $\mathbf{v_{i+1}} = \text{vec}(V_{i+1})$, and vec is a linear operator, we can vec^{-1} (a) to obtain:

$$J^k = \sum_{s=0}^{k} \lambda^{k-s} V_{s+1} \tag{3.17}$$

Since all V_p have non-zero entries in distinct positions, entry (i,j) depends solely on the coefficient of V_p where $p = j-i+1$. From (a) we know the V_p coefficient is $\binom{k}{p-1}\lambda^{k-(p-1)}$. Since $p = j-i+1$ we can substitute this into the expression for V_p in order to find the (i,j) entry. Thus the (i,j) entry equals:

$$\binom{k}{j-i}\lambda^{k-(j-i)} \tag{3.18}$$

Using the same logic, we can show equivalence in the reverse direction.

□

Lemma 3.9. *If J is a $n \times n$ Jordan block, then the n equivalence classes for the rows of $\Psi(J)$ are $r_1, r_2, ..., r_n$ with*

$$r_k = \{R_{(k-1)n+1+l(n+1)} | 0 \leq l \leq n-k\}$$

for $1 \leq k \leq n$ where R_j is the jth row of $\Psi(J)$.

Proof. Since $rank\Psi(J) = n$, there are n linearly independent rows of $\Psi(J)$. By Theorem 3.8 part (d), the (i,j)th entry of J^m is $\binom{m}{j-i}\lambda^{m-(j-i)}$ for $0 \leq m \leq n-1$. (By convention, this is 0 if $j-i<0$ or if $j-i>m$.) Since powers of J are upper triangular matrices, whenever $j<i$, the entry (i,j) is 0. Hence, all the rows of $\Psi(J)$ corresponding to the entries when $j<i$ in powers of J are rows of zeros. By definition of $\Psi(J)$ and by the vec ordering bijection between the entry (i,j) of J and the $(i+(j-1)n)$ row of $\Psi(J)$, the (i,j)th entry of J^m is the same as the $(i+(j-1)n, m+1)$ entry in $\Psi(J)$. Hence, each row corresponding to (i,j) of J is of the form:

$$R_{i+(j-1)n} = \begin{bmatrix} \binom{0}{j-i}\lambda^{-(j-i)} & \binom{1}{j-i}\lambda^{1-(j-i)} & \cdots & \binom{n-1}{j-i}\lambda^{n-1-(j-i)} \end{bmatrix}.$$

Then, the n rows in $\Psi(J)$ corresponding to diagonal entries, $i=j$, are:

$$R_{i+(i-1)n} = \begin{bmatrix} 1 & \lambda & \lambda^2 & \cdots & \lambda^{n-1} \end{bmatrix}$$

The $n-1$ rows in $\Psi(J)$ corresponding to entries when $j=i+1$ are:

$$R_{i+((i+1)-1)n} = \begin{bmatrix} 0 & 1 & \binom{2}{1}\lambda & \cdots & \binom{n-1}{1}\lambda^{n-2} \end{bmatrix}$$

The $n-2$ rows in $\Psi(J)$ corresponding to entries when $j=i+2$ are:

$$R_{i+((i+2)-1)n} = \begin{bmatrix} 0 & 0 & 1 & \cdots & \binom{n-1}{2}\lambda^{n-3} \end{bmatrix}$$

⋮

The 1 row in $\Psi(J)$ corresponding to the entry when $j = i + n - 1$ is:

$$R_{i+((i+n-1)-1)n} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \binom{n-1}{n-1}\lambda^0 \end{bmatrix}$$

Each of the n^2 rows of $\Psi(J)$ must be in one of these forms or it is a row of zeros. Note that there are n possible forms for the nonzero rows.

The first equivalence class, r_1, when $j = i$, consists of the rows of $\Psi(J)$ of the form: $R_{i+(i-1)n}$ for $1 \leq i \leq n$ which can be rewritten as $R_{(l+1)+((l+1)-1)n}$ for $0 \leq l \leq n - 1$.

The second equivalence class, r_2, when $j = i + 1$, consists of the rows of $\Psi(J)$ of the form: $R_{i+((i+1)-1)n}$ for $1 \leq i \leq n-1$ which can be rewritten as $R_{(l+1)+((l+2)-1)n}$ for $0 \leq l \leq n - 2$.
The third equivalence class, r_3, when $j = i+2$, consists of the rows of $\Psi(J)$ of the form: $R_{i+((i+2)-1)n}$ for $1 \leq i \leq n - 2$ which can be rewritten as $R_{(l+1)+((l+3)-1)n}$ for $0 \leq l \leq n - 3$.
We can continue this to get that the nth equivalence class, r_n, when $j = i+n-1$, consisting of the row of $\Psi(J)$ of the form: $R_{i+((i+n-1)-1)n}$ for $1 \leq i \leq 1$ which can be rewritten as $R_{(l+1)+((l+n)-1)n}$ for $0 \leq l \leq 0$.
Then each equivalence class is of the form:

$$r_k = \{R_{(l+1)+((l+k)-1)n} | 0 \leq l \leq n - k\}.$$

But $R_{(l+1)+((l+k)-1)n} = R_{(k-1)n+1+l(n+1)}$. Hence, each equivalence class is of the form:

$$r_k = \{R_{(k-1)n+1+l(n+1)} | 0 \leq l \leq n - k\} \text{ for } 1 \leq k \leq n.$$

□

Lemma 3.10. *If you take an element from each of the n equivalence classes, they form a linearly independent set.*

Proof. From Theorem 3.8 part (c), we know the ith entry in each row is $\binom{i-1}{k-1}\lambda^{i-k}$ for $1 \leq i \leq n$ where k is the kth equivalence class for $1 \leq k \leq n$, and if $i < k$, the ith entry is 0. Suppose we take one row from each of the n equivalences. Using $\binom{i-1}{k-1}\lambda^{i-k}$, the row taken from r_1 has a 1 in the first entry and nonzero entries everywhere else. The row taken from r_2 has a 0 in the first entry, a 1 in the second entry and nonzero entries everywhere else. Similarly, the row taken from r_3 has 0's in the first two entries, a 1 in the third entry and nonzero entries everwhere else. This goes on. The row taken from r_{n-1} has 0's in the first $n - 2$ entries, a 1 in the $(n - 1)$ entry, and a nonzero value in the last entry. Finally, the row taken from r_n will have 0's in the first $(n-1)$ entries and a 1 in the last entry. These n rows form an upper triangular matrix with 1's in the diagonal. Clearly, this matrix is invertible. Hence, the rows form a linearly independent set. □

Theorem 3.11. *Let J be $n \times n$ Jordan block. An admissible pattern α is of the form*

$$\alpha = \{(a_1, a_1), (a_2, a_2 + 1), ..., (a_{n-1}, a_{n-1} + (n - 2)), (a_n, a_n + (n - 1))\}$$

for $1 \leq a_k \leq n - k + 1$ and $1 \leq k \leq n$ iff rank($\Psi(J)_\alpha$)=n.

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Proof. Given $a_1, a_2, ..., a_n$ with $1 \leq a_k \leq n-k+1$, the specified entries of an α partial matrix X are $(a_1, a_1), (a_2, a_2+1), ..., (a_{n-1}, a_{n-1}+(n-2)), (a_n, a_n+(n-1))$. Each ordered pair $(a_k, a_k+(k-1)) \in \alpha$ corresponds to the specified entry $(a_k + (a_k + k - 2)n, 1)$ in vec(X) for $1 \leq k \leq n$. Hence, if we have $\Psi(J)\overrightarrow{C} = \text{vec}(X)$ where $\overrightarrow{C} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ for some constants $c_1, c_2, \cdots, c_n$ and n specified entries of vec$X$, it would result in n equations and n unknowns. Now, let $R_1, R_2, ..., R_{n^2}$ be the rows of $\Psi(J)$. Then each specified entry in X corresponds to an equation

$$R_{(a_k+(a_k+k-2)n)} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \text{value in } (a_k, a_k + (k-1))$$

Let $r_1, r_2, ..., r_n$ be the equivalence relations between the rows of $\Psi(J)$ as in Lemma for the Rows of $\Psi(J)$ 1. It follows that each $R_{a_k+(a_k+k-2)n} \in r_k$ for $1 \leq k \leq n$. By Lemma for the Rows of $\Psi(J)$ 2, the n rows are linearly independent. In other words, the $rank({}_\alpha\Psi(J)) = n$. □
Conversely, suppose rank$({}_\alpha\Psi(J)) = n$. Then the specified entry locations in α correspond to n linearly independent rows. We know the n equivalence classes between the rows of $\Psi(J)$. If we take more than one row from one of the r_k's in Lemma 3.1 for the n rows we choose, we'll have at most $n-1$ linearly independent rows. In order to get n linearly independent rows, we must take a row from each of the row equivalence classes so Lemma 3.2 holds. We know

$$R_{(k-1)n+1+l(n+1)} \in r_k$$

for $1 \leq k \leq n$ and $0 \leq i \leq n-k$. $R_{(k-1)n+1+l(n+1)}$ corresponds to the specified entry $(l+1, l+k)$ in the partial matrix X. Now, let $a_k = l+1$ so that the specified entries of X are of the form $(a_k, a_k + k - 1)$ for $1 \leq a_k \leq n-k+1$. Therefore, α is of the form $\alpha = \{(a_1, a_1), (a_2, a_2 + 1), ..., (a_{n-1}, a_{n-1} + (n-2)), (a_n, a_n + (n+1))\}$ for $1 \leq a_k \leq n-k+1$. □

3.2. Matrix Equation.

Lemma 3.12. *Show that* $S_i \notin span\{S_0, S_1, ..., S_{i-1}, S_{i+1}, ..., S_{n-1}\}$

Proof. We know that by definition $S_i = \{c_{1+(n-1-i)n+j(n+1)} | \, 0 \leq j \leq i\}$ Each row of $\Omega(A)$ has at most two nonzero entries. If it just has one nonzero entry, the corresponding column of this entry in S_i couldn't be a linear dependence of the other columns of S_k for some $k \in \{0, 1, ..., k-1, k+1, ..., n-1\}$. If it has two nonzero entries, we have that both columns corresponding to the entries are inside of S_i, for some i, because one of them is from $-\mathbb{I}_n$ and the other one is from $A - \lambda\mathbb{I}$. The number of columns between them is n, and this is the characterization for S_i. As in the 1 □

$$\Omega(A) = \left(\begin{array}{c|c|c|c} \ddots & & & \\ \hline & A - \lambda\mathbb{I} & \mathbf{0} & \\ \hline & -\mathbb{I} & A - \lambda\mathbb{I} & \\ \hline & & & \ddots \end{array}\right)$$

Lemma 3.13. *Show that* $rank(span\{S_i\}) = i$

$$
=\left[\begin{array}{c|c|c|c}
\ddots & & & \\ \hline
 & \begin{matrix}0&1&&&\\&0&1&&\\&&\ddots&\ddots&\\&&&0&1\\&&&&0\end{matrix} & & \\ \hline
 & \begin{matrix}-1&&&&\\&-1&&&\\&&\ddots&&\\&&&-1&\\&&&&-1\end{matrix} & \begin{matrix}0&1&&&\\&0&1&&\\&&\ddots&\ddots&\\&&&0&1\\&&&&0\end{matrix} & \\ \hline
 & & & \ddots
\end{array}\right]
$$

FIGURE 1. And each submatrix here is an element of space $\mathbb{M}_{n\times n}$

Proof. By definition, S_i is the set that contains the linear dependencies between the columns of $\Omega(A)$. We know the relationship between the columns. Hence, the elements in S_i are of the form:

$$c_{1+(1-i-j)n} = \sum_{j=1}^{i} c_{1+(n-1-j)n+j(n+1)} \; for 0 \leq j \leq i \tag{3.19}$$

Therefore $rank(span\{S_i\}) \leq i$. If we look at the rows with two nonzero entries, the corresponding columns of the entries are linear dependent. If we remove a column, we get a row with only one nonzero entry. Let's say this column is c_p. WLOG let's assume the fist column of S_i was removed and c_p stays. Suppose $rank(span\{S_i\}) < i$, we would get

$$\sum_{j=1}^{i} k_i c_{1+(n-1-j)n+j(n+1)} = 0$$

with some k_i being nonzero, but we have a row with just one nonzero entry. The entry is in the corresponding column c_p, and c_p is one element of the summation, for $1 \leq p \leq j$. However, $k_p = 0$ implies $k_{p+(n+1)}$ or $k_{p-(n+1)}$ are zero, as in 2. Finally, we get $k_j = 0$ for all $0 \leq j \leq i$ which is a contradiction. Hence all the columns in S_i are linearly independent and $rank(span\{S_i\}) = i$ □

Lemma 3.14. *$rank(\Omega(J)) = rank(\Omega(J_{\alpha^c}))$ where $\alpha = \{a_0, a_1, \cdots, a_{n-1}\}$ and $c_{a_i} \in S_i$ for all $0 \leq i \leq n-1$.*

Proof. We have $c_{a_0} = \overrightarrow{0}$ and by Lemma 3.5 $c_{a_0}, c_{a_1}, ..., c_{a_{n-1}}$ are linearly independent. then the remaining columns of $\Omega(A)$, denoted $c_{a'_i} \in \alpha^c$, are linearly independent and the size of α^c is $n^2 - n$. We know that $rank(\Omega(J)) = n^2 - n$ and we get $rank(\Omega(J)_{\alpha^c}) = n^2 - n$
$\therefore\ rank(\Omega(J)) = rank(\Omega(J)_{\alpha^c})$ □

$$\left|\begin{array}{cccccccc}
\ddots & & & & & & & \\
 & -1 & \overbrace{\cdots}^{n} & 1 & & & & \\
 & & & n\left\{\vdots\right. & & & & \\
 & & & -1 & \overbrace{\cdots}^{n} & 1 & & \\
 & & & & & \vdots & & \\
 & & & & & \cdots & -1 \;\overbrace{\cdots}^{n} & 1 \\
 & & & & & & & \ddots
\end{array}\right|$$

FIGURE 2. Number of columns between the entries of a row with two entries nonzero

3.3. Graph Theoretic.

Definition 3.15. The graph of alpha, denoted $G(\alpha)$, is $G(\alpha) = (V, E, M)$ where $V = \{1, \cdots, n\}$, $E = (e_1, \cdots, e_{n-1})$ where $e_i \in E_i$, and $M = \{v\}$, where $v \in V$.

Definition 3.16. Consider the following subsets of edges in a directed graph with n vertices:
$E_0 = V$
$E_1 = \{(1, 2), (2, 3), (3, 4), \cdots, (n-1, n)\}$
$E_2 = \{(1, 3), (2, 4), (3, 5), \cdots, (n-2, n)\}$
$E_3 = \{(1, 4), (2, 5), (3, 6), \cdots, (n-3, n)\}$
$\vdots$
$E_{n-2} = \{(1, n-1), (2, n)\}$
$E_{n-1} = \{(1, n)\}$

Lemma 3.17. *Let α be a pattern for an $n \times n$ Jordan Block. The pattern has the form $\alpha = \{(a_1, a_1), (a_2, a_2+1), \cdots, (a_{n-1}, a_{n-1}+(n-2)), (1, n)\}$ for $1 \leq a_i \leq n-i+1$ if and only if the graph corresponding to alpha has the form $G(\alpha) = (V, E, M)$ with $V = \{1, \ldots, n\}$, $E = \{e_1, \ldots, e_{n-1}\}$ with $e_i \in E_i$, and M=$\{v\}$ with $v \in V$.*

Proof. $3 \Leftrightarrow 4$
(i.) $\Rightarrow$ | Suppose $\alpha = \{(a_1, a_1), (a_2, a_2+1), \cdots, (a_{n-1}, a_{n-1}+(n-2)), (1, n)\}$ for $1 \leq a_i \leq n-i+1$. By definition, $G(\alpha) = (V, E, M)$ is a marked, directed graph where $V = \{1, \cdots, n\}$, $E = \{(i, j) | i, j \in \alpha, i < j\}$, and $M = \{i | (i, i) \in \alpha\}$. Also, by definition, E is the set of subsets in a directed graph with n vertices. $E_0 = v$, $E_1 = \{(1, 2), (2, 3), \cdots, (n-1, n)\}$, $E_2 = \{(1, 3), (2, 4), \cdots, (n-2, n)\}$, $\cdots$, $E_{n-2} = \{(1, n-1), (2, n)\}$, $E_{n-1} = \{(1, n)\}$. The diagonal ordered pair $(a_2, a_2+1) \in E_1$. The next off-diagonal ordered pair $(a_3, a_3+2) \in E_2, \cdots$. Then, the ordered pair $(a_{n-1}, a_{n-1}+(n-2)) \in E_{n-2}$, and the ordered pair $(1, n) \in E_{n-1}$. Therefore, $G(\alpha) = (\{1, \cdots, n\}, \{e_1, \cdots, e_{n-1}\}, \{v\})$, where $e_i \in E_i$ for $1 \leq i \leq n-1$ and $v \in \{1, \cdots, n\}$.
(ii.) $\Leftarrow$ | Suppose $G(\alpha) = (\{1, \cdots, n\}, \{e_1, \cdots, e_{n-1}\}, \{v\})$, where $e_i \in E_i$ for $1 \leq i \leq n-1$ and $v \in \{1, \cdots, n\}$. By definition, of $E_t = \{(i, i+j)\}$ where $1 \leq t \leq n-1$, $1 \leq i \leq n-t$, and $1 \leq j \leq n-1$. Letting $1 \leq a_i \leq n-i+1$ and $1 \leq j \leq n-1$, generates all elements of E_t. Therefore, $(a_i, a_i+j) \in E_j$ where $1 \leq a_i \leq n-i+1$ and $1 \leq j \leq n-1$. Thus, $\alpha = \{(a_1, a_1), (a_2, a_2+1), \cdots, (a_{n-1}, a_{n-1}+(n-2)), (1, n)\}$ for $1 \leq a_i \leq n-i+1$. $\square$

3.4. **Characterization of admissible patterns for one Jordan block.** We collect the results in this section in the following result.

Theorem 3.18. *Let J be an $n \times n$ Jordan block. Then, the following are equivalent:*

(1) *α is an maximal admissible pattern*

(2) *Rank($\Omega(J)_{\alpha^c}$) $= n^2 - n$*

(3) *$\alpha = \{(a_1, a_1), (a_2, a_2 + 1), \cdots, (a_{n-1}, a_{n-1} + (n-2)), (1, n)\}$ for $1 \leq a_i \leq n - i + 1$*

(4) *$G(\alpha) = (V, E, M)$ with $V = \{1, \ldots, n\}$, $E = \{e_1, \ldots, e_{n-1}\}$ with $e_i \in E_i$, and M=$\{v\}$ with $v \in V$.*

(5) *Rank($_\alpha\Psi(J)$) $= n$*

Proof. Therefore it is true by consequence of being true. □

4. Building Patterns

Goal for this section is to describe how to build new admissible patterns. We have three methods for building new admissible patterns: direct sums of patterns, permutation similarity, and using the nullspace of $\Omega(A)$.

4.1. **Direct sum of patterns.**

Definition 4.1. **??** Let α be a pattern of specified entries for an $n_1 \times n_1$ matrix and β be a pattern of specified entries for an $n_2 \times n_2$ matrix.
$\alpha = \{(a_1, b_1), (a_2, b_2), \cdots, (a_k, b_k) | 1 \leq a_i \leq n_1, 1 \leq b_j \leq n_1 \text{ for } 1 \leq i, j \leq k\}$
$\beta = \{(c_1, d_1), (c_2, d_2), \cdots, (c_m, d_m) | 1 \leq c_i \leq n_2, 1 \leq d_j \leq n_2 \text{ for } 1 \leq i, j \leq m\}$
The directed sum, denoted $\alpha \oplus \beta$, is
$\alpha \oplus \beta = \alpha \cup \{(c_1 + n_1, d_1 + n_1), (c_2 + n_1, d_2 + n_1), \cdots, (c_m + n_1, d_m + n_1)\}$.

Lemma 4.2. *Let α be an admissible pattern for a $m \times m$ matrix A and β be an admissible pattern for a $n \times n$ matrix B. The partial matrix pattern $\alpha \oplus \beta$ is an admissible pattern for the matrix $A \oplus B$.*

Proof. Since α is an admissible pattern for A, then an α-partial matrix X of the same size commutes with A. Similarly, since β is an admissible pattern for B, then a β-partial matrix Y of the same size commutes with B. By [**?**, pg. 24], if A commutes with X and B commutes with Y, then $A \oplus B$ commutes with $X \oplus Y$. Hence, the pattern of the partial matrix $X \oplus Y$ is admissible. By Definition **??**, this pattern is $\alpha \oplus \beta$. □

We can obtain the graphs for these direct sum patterns with a disjoint union of the component graphs.

Definition 4.3. Let G_1 and G_2 be marked directed graphs. $G_1 = (V_1, E_1, M_1)$, where $V_1 = \{1, \cdots, n_1\}$, and $G_2 = (V_2, E_2, M_2)$, where $V_2 = \{1, \cdots, n_2\}$. The fusion of G_1 and G_2, denoted $G_1 \uplus G_2$, is $G_1 \uplus G_2 = (V', E', M')$, where
$V' = V_1 \cup \{i + n_1 | i \in V_2\}$, $E' = E_1 \cup \{(i + n_1, j + n_1) | (i, j) \in E_2\}$, and $M' = M_1 \cup \{i + n_1 | i \in M_2\}$.

Lemma 4.4. $G(\alpha) \uplus G(\beta) = G(\alpha \oplus \beta)$

Proof. Let α be a pattern for an $n_1 \times n_1$ matrix where $\alpha = \{(a_1, b_1), (a_2, b_2), \cdots, (a_k, b_k)\}$, and let β be a pattern for an $n_2 \times n_2$ matrix where $\beta = \{(c_1, d_1), (c_2, d_2), \cdots, (c_m, d_m)\}$.
The direct sum, $\alpha \oplus \beta = \alpha \cup \{(c_1 + n_1, d_1 + n_1), (c_2 + n_1, d_2 + n_1), \cdots, (c_k + n_1, d_k + n_1)\}$.
Then $G(\alpha \oplus \beta) = (V, E, M)$, where the vertex set $V = \{1, 2, \cdots, n_1 + n_2\}$, the edge set $E = \{(a_1, b_1), \cdots, (a_k, b_k), (c_1 + n_1, d_1 + n_1), \cdots, (c_m + n_1, d_m + n_1)\}$, and the marked set $M = \{(1, 1), (2, 2), \cdots, (n_1, n_1), \cdots, (n_2 + n_1, n_2 + n_1)\}$.
The fusion, $G(\alpha) \uplus G(\beta) = (V', E', M')$, where the vertex set $V' = \{1, 2, \cdots, n_1 + n_2\}$, the edge set $E' = \{(a_1, b_1), \cdots, (a_k, b_k), (c_1 + n_1, d_1 + n_1), \cdots, (c_m + n_1, d_m + n_1)\}$, and the marked set $M' = \{(1, 1), (2, 2), \cdots, (n_1, n_1), \cdots, (n_2 + n_1, n_2 + n_1)\}$.
Since $V = V'$, $E = E'$, and $M = M'$, $G(\alpha \oplus \beta) = G(\alpha) \uplus G(\beta)$. □

4.2. **Permutations of patterns.**

Definition 4.5. A $n \times n$ **permutation matrix** Q is obtained by permuting the rows of the $n \times n$ Identity Matrix. Every column and every row of Q has exactly one nonzero entry, that entry is a 1.

Definition 4.6. A **permutation** σ associated with a permutation matrix Q^{-1} is defined as $\sigma(j) = i$ if the entry $Q^{-1}_{i,j}$ is 1.

Note: The entry $(\sigma(i), \sigma(j))$ in an $n \times n$ matrix $A = Q^{-1}JQ$ is the same as the entry (i, j) in a $n \times n$ Jordan block J where σ is the associated with Q^{-1} because of what Q^{-1} does to the rows and columns of J.

Theorem 4.7. *Suppose* $\alpha = \{(a_1, a_1), (a_2, a_2 + 1), ..., (a_n, a_n + (n - 1))\}$ *is an admissible pattern for a* $n \times n$ *Jordan block, J. Then* $\beta = \{(\sigma(a_1), \sigma(a_1)), (\sigma(a_2), \sigma(a_2 + 1)), ..., (\sigma(a_n), \sigma(a_n + (n-1)))\}$ *is an admissible pattern for a* $n \times n$ *matrix A that is permutation equivalent to J. In other words,* $A = Q^{-1}JQ$ *where Q is* $n \times n$ *permutation matrix and* σ *is the permutation associated with* Q^{-1}.

Proof. Suppose α is an admissible pattern for J. By the Classification Theorem, the rank ${}_\alpha\Psi(J) = n$. Since Q is a permutation, it preserves all the entry values of matrices, but not their positions. Therefore, since $A = Q^{-1}JQ$, $A^k = Q^{-1}J^kQ$ for all $k \geq 0$. By definition, $\Psi(J) = \begin{bmatrix} | & | & | & | & | \\ J^0 & J^1 & J^2 & \cdots & J^{n-1} \\ | & | & | & | & | \end{bmatrix}$ and $\Psi(A) = \begin{bmatrix} | & | & | & | & | \\ A^0 & A^1 & A^2 & \cdots & A^{n-1} \\ | & | & | & | & | \end{bmatrix}$. Now, the entry (i, j) in J^k is the same as the entry $(\sigma(i), \sigma(j))$ in A^k for $k \geq 0$. So for each entry $(a_i, a_i + (i-1)) \in \alpha$,

$$R_{a_i + (a_i + i - 2)n} \text{ in } \Psi(J) = R_{\sigma(a_i) + (\sigma(a_i + (i-1)) - 1)n} \text{ in } \Psi(A) \text{ for } 1 \leq i \leq n$$

In other words,

$${}_\alpha\Psi(J) = \begin{bmatrix} - & R_{a_1 + (a_1 - 1)n} & - \\ - & R_{a_2 + (a_2)n} & - \\ \vdots & \vdots & \vdots \\ - & R_{a_n + (a_n + n - 2)n} & - \end{bmatrix} = \begin{bmatrix} - & R_{\sigma(a_1) + (\sigma(a_1) - 1)n} & - \\ - & R_{\sigma(a_2) + (\sigma(a_2 + 1) - 1)n} & - \\ \vdots & \vdots & \vdots \\ - & R_{\sigma(a_n) + (\sigma(a_n + (n-1)) - 1)n} & - \end{bmatrix} =_\beta \Psi(A)$$

In particular, rank ${}_\alpha\Psi(J)$=rank ${}_\beta\Psi(A)$=n. By Theorem 6.2 (3), β is an admissible pattern for A. □

Corollary 4.8. *If $B \oplus A = Q^T(A \oplus B)Q$ and $\alpha \oplus \beta$ is an admissible pattern for $A \oplus B$, then γ is an admissible pattern for $B \oplus A$ with $(\sigma(i), \sigma(j)) \in \gamma$ for each $(i, j) \in \alpha \oplus \beta$ where σ is the permutation associated with Q^T.*

4.3. **Obtaining pattern from the nullspace of $\Omega(A)$.**

Definition 4.9. The reduced basis, $\mathbf{b_1}...\mathbf{b_q}$, is the basis for a space which is formed by transposing the matrix of all the basis vectors, row reducing it, and transposing them back into a column vectors.

Lemma 4.10. *Let $\mathbf{b_1}...\mathbf{b_q}$ be a basis for the nullspace of matrix A and let k be a non-zero entry of a basis vector, $\mathbf{b_1}$, of the nullspace of matrix A. Then the span of the basis vectors $\mathbf{b_2}...\mathbf{b_p}$ with entry k removed from each is equal to the nullspace of matrix A with the k column removed. That is to say:*

$$\text{span}\{_{k^C}\mathbf{b_2}..._{k^C}\mathbf{b_q}\} = \text{null}(A_{k^C})$$

Proof. Proof goes here. □

Lemma 4.11. *An $m \times n$ matrix with all rows being linearly independent in reduced row echelon form will have exactly one non-zero entry in at least m columns for $m \leq n$.*

Proof. Since all the matrix's rows are linearly independent and $m \leq n$, the matrix will have exactly m pivot positions. By the properties of reduced row echelon form, each pivot column will have the property that exactly one entry is 1 and all other entries in that column are 0. Since there are m pivot positions, this means that in reduced row echelon form there will be at least m columns with exactly 1 non-zero entry. □

Let A be any $n \times n$ matrix and let the rank of $\Omega(A)$ be p. Then the dimension of the nullspace of $\Omega(A)$ is $n^2 - p$ and the reduced basis for the nullspace is $\mathbf{b_1}...\mathbf{b_{n^2-p}}$. Let a_1 be any non-zero entry in $\mathbf{b_1}$ and let a_i be any nonzero entry of $\mathbf{b_i}$ such that a_i is not equal to any previous a. Let $\alpha = \{a_1...a_i\}$. Then the following lemma is true:

Lemma 4.12.

$$\text{rank}(\Omega(A_{\alpha^C})) = \text{rank}(\Omega(A)) = n^2 - p$$

Proof. Let A be any $n \times n$ matrix and let the rank of $\Omega(A)$ be p. Then the dimension of the nullspace of $\Omega(A)$ is $n^2 - p$ and the reduced basis for the nullspace is $\mathbf{b_1}...\mathbf{b_{n^2-p}}$. Let a_1 be any non-zero entry in $\mathbf{b_1}$ and let a_i be any nonzero entry of $\mathbf{b_i}$ such that a_i is not equal to any previous a. Let $\alpha = \{a_1...a_i\}$. Each basis in the reduced nullspace shows a dependence relationship from the columns of $\Omega(A)$. Thus by choosing a_1 to correspond to a non-zero entry in $\mathbf{b_1}$ the removal of a_1 from $\Omega(A)$ will not reduce the rank since a_1 is part of a linearly dependent set. By Lemma 4.10, $\text{span}\{_{a_1^C}\mathbf{b_2}..._{k^C}\mathbf{b_{n^2-p}}\} = \text{null}(\Omega(A)_{a_1^C})$. Thus the removal of any vector does not affect the linear dependencies in the other basis vectors. By the same reasoning the rank held after choosing a_1, choosing $a_2...a_{n^2-p}$ will also not reduce the rank since the reduced basis vectors maintain the same linear dependencies after removal of the previous a. Thus the same pattern of choosing one non-zero entry from each reduced basis vector will not reduce the rank. By Lemma 4.11 we know that since our reduced basis is in transposed reduced row echelon form, there will be at most one non-zero entry in at least $n^2 - p$ rows. Since there are n^2 entries in each basis vector and only $n^2 - p$ basis vectors, since $n^2 - p \leq n^2$ there will always be enough entries for a distinct entry to be chosen from each column. Thus each column will have at least one entry distinct from all the other columns so it will always be possible to choose an entry a_i such that a_i is not equal to any previous a. $\therefore \text{rank}\Omega(A_{\alpha^C}) = \Omega(A) = n^2 - p$. □

Theorem 4.13. *There exists a maximal admissible pattern for any square $n \times n$ matrix A which can be constructed from the reduced basis for the nullspace of the omegamatrix.*

Proof. Let A be any $n \times n$ matrix and the reduced basis for the nullspace is $\mathbf{b_1}...\mathbf{b_{n^2-p}}$. Let a_1 be any non-zero entry in $\mathbf{b_1}$ and let a_i be any nonzero entry of $\mathbf{b_i}$ such that a_i is not equal to any previous a. Let $\alpha = \{a_1...a_i\}$ where $1 \leq i\ leqn^2 - p$. Then by Lemma 4.12 the rank of $\Omega(A_{\alpha^C})$ is equal to the rank of $\Omega(A)$ and thus by Lemma (Buhls) α is an admissible pattern. When $i = n^2 - p$, α is a maximal admissible pattern for A since by Lemma 4.10 and Lemma 4.12 it follows that since there are $n^2 - p$ vectors in the nullspace, the nullspace of $\Omega(A_{\alpha^C})$ will have a dimension of zero since the nullspace is equal to the span of the basis vectors $\mathbf{b_1}...\mathbf{b_{n^2-p}}$ with $n^2 - p$ basis vectors removed. Thus the columns of $\Omega(A_{\alpha^C})$ are linearly independent and therefore by definition, when $i = n^2 - p$ α is maximally admissible since there is no way to choose more specified entries and have rank($\Omega(A_{\alpha^C})$) = rank($\Omega(A)$) still hold. □

Theorem 4.14. *Let J be an $k \times k$ Jordan block, and A a $n \times n$ matrix in Jordan canonical form, and let γ be an admissible pattern for J and α be an admissible pattern for A. Then, the following are equivalent:*

(1) $\alpha \oplus \gamma$ *is an admissible pattern for* $A \oplus J$

(2) *Rank*$(\Omega(A \oplus J)_{\alpha \oplus \gamma^c}) = (n+k)^2 - (n+k)$

(3) $\alpha \oplus \gamma = \{(a_1, a_1), (a_2, a_2+1), \cdots, (a_{n-1}, a_{n-1}+(n-2)), (1, n), (b_1+n, b_1+n), ((b_2+n, b_2+1+n), \cdots, (b_{k-1}+n, b_{k-1}+(k-2)+n), (1+n, k+n)$

(4) $G(\alpha) \uplus G(\gamma)$ *is a directed, acyclic graph with* $n-1$ *edges. Its edges are of the form* (i, j) *such that* $i = j$ *or* $i < j$.

(5) *Rank*$({}_{\alpha \oplus \gamma}\Psi(A \oplus J)) = n + k$

5. Maximal Admissible Patterns for Jordan Canonical Form

If a $n \times n$ matrix A is nonderogatory, the space of matrices that commute with A is n dimensional, and the size of a maximal admissible pattern for A is n. In this section, we discuss how to generalize our results for one Jordan block to obtain maximal admissible patterns for matrices in Jordan canonical form without that assumption that these matrices are nonderogatory.

Lemma 5.1. *Let J be a Jordan block with eigenvalue λ and size $n \times n$. The relationship between the columns of $\Omega(J)$ is*

$$\begin{aligned} c_{1+(n-1)n} &= 0 \\ c_{1+(n-2)n} &= c_{1+(n-1)+1} \\ &\vdots \\ c_{1+n} &= \sum_{i=1}^{n-2} c_{1+n+i(n+1)} \\ c_1 &= \sum_{i=1}^{n-1} c_{1+i(n+1)} \end{aligned}$$

or on a condensed formula:

$$\sum_{i=0}^{n-1-j} c_{1+jn+i(n+1)} = 0 \tag{5.1}$$

for j such as $0 \leq j \leq n-1$.

By induction.
• Case $n = 1$
In this case:

$$\begin{aligned} j &= 0 \\ J &= \left[\begin{array}{c} a_{11} \end{array} \right] \\ \Omega(J) &= \left[\begin{array}{c} 0 \end{array} \right] \end{aligned}$$

We apply the formula and we have:

$$c_1 = \sum_{i=1}^{0} c_{1+i(2)} = 0$$

• Assume the case $n = k$ true:

$$c_{1+jk} = \sum_{i=1}^{k-1-j} c_{1+jk+i(k+1)} \tag{5.2}$$

$$\begin{aligned} \Omega(J) &= \mathbb{I} \otimes J - J^T \otimes \mathbb{I} \\ &= \begin{bmatrix} J-\lambda\mathbb{I} & & & & \\ -\mathbb{I} & J-\lambda\mathbb{I} & & & \\ & -\mathbb{I} & \ddots & & \\ & & \ddots & J-\lambda\mathbb{I} & \\ & & & -\mathbb{I} & J-\lambda\mathbb{I} \end{bmatrix} \end{aligned}$$

The following pattern is the same inside of $\Omega(J)$
$\begin{bmatrix} J-\lambda\mathbb{I} & 0 \\ -\mathbb{I} & J-\lambda\mathbb{I} \end{bmatrix}$

This is:

$$\left[\begin{array}{ccccc|ccccc}
0 & 1 & & & & & & & & \\
 & 0 & 1 & & & & & & & \\
 & & \ddots & \ddots & & & & & & \\
 & & & 0 & 1 & & & & & \\
 & & & & 0 & 0 & & & & \\
\hline
-1 & & & & & 0 & 1 & & & \\
 & -1 & & & & & 0 & 1 & & \\
 & & \ddots & & & & & \ddots & \ddots & \\
 & & & -1 & & & & & 0 & 1 \\
 & & & & -1 & & & & & 0
\end{array}\right]$$

• Let's prove the case $n = k+1$

The columns $i\,(k+1)$ for $1 \leq i \leq k$ are linear independent of others columns of $\Omega\,(J)$ because the entries $\Omega\,(J)_{(i+1)(k+1),i(k+1)}$ for $1 \leq i \leq k$ are the only one entries in the row $(i+1)\,(k+1)$ different of 0. Given the linear dependencies from $\Omega\,(J_{k\times k})$ we need to change into the $\Omega\left(J_{(k+1)\times(k+1)}\right)$, and we just change k by $k+1$. Finally we see the linear dependencies of the last $k+1$ columns of $\Omega\left(J_{(k+1)\times(k+1)}\right)$:

$$\begin{aligned}
c_{1+([k+1]-1)(k+1)} &= 0 \\
c_{1+([k+1]-2)(k+1)+[k+1]+1} &= c_{1+([k+1]-2)(k+1)} \\
&\vdots \\
c_{1+(k+1)+(k-1)([k+1]+1)} &= c_{1+(k+1)} - \left[c_{1+(k+1)+[k+1]+1} + \cdots + c_{1+(k+1)+k([k+1]+1)}\right] \\
c_{1+k([k+1]+1)} &= c_1 - \left[c_{1+[k+1]+1} + \cdots + c_{1+(k-1)([k+1]+1)}\right]
\end{aligned}$$

This is

$$c_{1+j(k+1)} = \sum_{i=1}^{k-j} c_{1+j[k+1]+i([k+1]+1)} \tag{5.3}$$

for $1 \leq j \leq k$

$$\Omega(J_{k+1,k+1}) = \left(\begin{array}{cccccc|cccccc}
0 & 1 & & & & & & & & & & \\
 & 0 & 1 & & & & & & & & & \\
 & & \ddots & \ddots & & & & & & & & \\
 & & & 0 & 1 & & & & & & & \\
 & & & & 0 & 1 & 0 & & & & & \\
 & & & & & 0 & 0 & 0 & & & & \\
\hline
-1 & & & & & & 0 & 1 & & & & \\
 & -1 & & & & & & 0 & 1 & & & \\
 & & -1 & & & & & & 0 & 1 & & \\
 & & & -1 & & & & & & 0 & 1 & \\
 & & & & -1 & & & & & & 0 & 1 \\
 & & & & & -1 & & & & & & 0
\end{array}\right)$$

And each submatrix here is an element of space $\mathbb{M}_{n\times n}$

■

Lemma 5.2. *Let* A *be a matrix of size n, and let J_i be Jordan blocks for $i \leq i \leq k$ such as* $A = \bigoplus_{i=1}^{k} J_i$.
Given a entry p $(1 \leq p \leq {m_i}^2)$ *in a Jordan block J_i, we get the corresponding column in $\Omega(A)$ as:*

$$c_p = c_{p+n\sum_{r=1}^{i-1} m_i + (t+1)\sum_{r=1}^{i-1} m_i + t\sum_{i+1}^{k} m_i}$$

with $m_i = dimJ_i$ and $t = \lfloor \frac{p-i}{m_i} \rfloor$[1]

Wlog we are going to work on J_i Jordan block.
The dimension of J_i is m_i. And we want to find the corresponding column on $\Omega(A)$ of an entry p such as c_p is from the relationship of linear dependence in J_i.
We consider all columns before the corresponding set of J_i, as $A - \lambda_i\mathbb{I}$ is a matrix of size n and according with the dimension of each $J_i, (1 \leq j \leq k)$ we have a set of $n(m_i)$ for its entries. We need add $n(m_1 + \cdots + m_{i-1})$ columns to reach the set where the diagonal has entries zero corresponding at J_i Jordan block, this is $n\sum_{r=1}^{i-1} m_i$.
For each entry in J_i we have a corresponding column in the matrix $\Omega(A)$, where each set $\sum_{r=1}^{k} m_i$ of columns denoted as T, we have a column from each Jordan block inside of each set like this. But we have the size of T equal to all columns on the Jordan blocks, so we need to add the corresponding columns of the $J_j : j \neq i$, for this, we can locate the column corresponding to p in J_i, we need to subtrack one unit to p(to get the number such as $0 \leq p \leq m_i$) and do the quotient of this difference and the size of m_i and this is t, but we need to add 1 to t to get the column, because the fist m_i can't be in the column zero. We can separate the set of columns $J_j : j \neq i$ as following sets: $T_1 = \{J_j | 1 \leq j \leq i-1\}$ and $T_2 = \{J_j | i+1 \leq j \leq m_i\}$, we need to add $t+1$ times the columns corresponding to Jordan block before at J_i and t times the columns corresponding to Jordan block after at J_i among the set of columns T corresponding at Jordan block J_i where the diagonal have entries equal to zero, in total we have to add $(t+1)\sum_{r=1}^{i-1} m_i + t\sum_{i+1}^{k} m_i$ columns to p to get the corresponding column on $\Omega(A)$

Lemma 5.3. *Let A be a matrix of size n, and let J_i be Jordan blocks for $1 \leq i \leq k$ such as* $A = \bigoplus_{i=1}^{k} J_i$. *with eigenvalues λ_i.*
If the eigenvalues λ_i and λ_j are such that $\lambda_i \neq \lambda_j$ for all $i \leq j$, the relationship of linear dependence of $\Omega(A)$ is the same as all J_i in the corresponding columns in $\Omega(A)$

Proof. We have in $\Omega(A)$ two different kind of matrix: $-\mathbb{I}$ and $A - \lambda_i\mathbb{I}$, for this reason (figure 3), we have in some place in the matrix entries equal to zero, corresponding at Jordan block J_i, others columns have entries different of zero in the diagonal and they are linearly independent, for this reason if we erase these columns and the rows corresponding at complement of J_i we get the matrix $\Omega(J_i)$ but this is for all i, we know the relationship between the columns in each J_i denoted as $\mathcal{J}_i$ and each $\mathcal{J}_i$ is linearly independent to $\mathcal{J}_j$ for all $i \neq j$ (figure 4). Using the lemma 5.1 we obtain the relationship between the columns for all J_i and using lemma 5.2 we obtain the corresponding columns on $\Omega(A)$ (figure 5). □

[1] $\lfloor\rfloor$ is the Step Function

$$A = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{bmatrix} = \begin{bmatrix} \lambda_1 & 1 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \ddots & 1 & & & & \\ & & & \lambda_1 & & & & \\ & & & & \ddots & & & \\ & & & & & \lambda_k & 1 & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & \ddots & 1 \\ & & & & & & & & \lambda_k \end{bmatrix}$$

FIGURE 3. Each subblock has a size $n \times m_i$

$$\Omega(A) = \begin{bmatrix} A-\lambda_1\mathbb{I} & & & & & & & & & & & & \\ -\mathbb{I} & \ddots & & & & & & & & & & & \\ & \ddots & \ddots & & & & & & & & & & \\ & & -\mathbb{I} & A-\lambda_1\mathbb{I} & & & & & & & & & \\ & & & & \ddots & & & & & & & & \\ & & & & & A-\lambda_i\mathbb{I} & & & & & & & \\ & & & & & -\mathbb{I} & \ddots & & & & & & \\ & & & & & & \ddots & \ddots & & & & & \\ & & & & & & & -\mathbb{I} & A-\lambda_i\mathbb{I} & & & & \\ & & & & & & & & & \ddots & & & \\ & & & & & & & & & & A-\lambda_k\mathbb{I} & & \\ & & & & & & & & & & -\mathbb{I} & \ddots & \\ & & & & & & & & & & & \ddots & \ddots \\ & & & & & & & & & & & -\mathbb{I} & A-\lambda_k\mathbb{I} \end{bmatrix}$$

FIGURE 4. Difference of the matrices, Each subblock in this matrix has size $n \times m_i$

6. FURTHER RESEARCH

[h] Future research can entail exploring the types of patterns that are admissible for matrices of a specific class such as upper triangular, nonsingular, singular, symmetric, etc. Future work can also be done on permutations of admissible patterns for other nonderogatory matrices, instead of just permutations of a Jordan block. In general, the process of classifying admissible patterns for any matrix can be simplified.

[h]

$$A - \lambda_i \mathbb{I} = \begin{bmatrix} \lambda_1 - \lambda_i & 1 & & & & & & & & & & \\ & \ddots & \ddots & & & & & & & & & \\ & & \ddots & 1 & & & & & & & & \\ & & & \lambda_1 - \lambda_i & & & & & & & & \\ & & & & \ddots & & & & & & & \\ & & & & & 0 & 1 & & & & & \\ & & & & & & \ddots & \ddots & & & & \\ & & & & & & & \ddots & 1 & & & \\ & & & & & & & & 0 & & & \\ & & & & & & & & & \ddots & & \\ & & & & & & & & & & \lambda_k - \lambda_i & 1 \\ & & & & & & & & & & \ddots & \ddots \\ & & & & & & & & & & & \ddots \quad 1 \\ & & & & & & & & & & -\mathbb{I} & \lambda_1 - \lambda_i \end{bmatrix}$$

FIGURE 5. We have on the diagonal entries equal to zero when the eigenvalues of the difference are the same

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